JOURNAL OF APPROXIMATION THEORY **83**, 1–40 (1995)

Generalized Orthogonality and Continued Fractions*

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Communicated by Doron S. Lubinsky

Received March 3, 1994; accepted in revised form August 2, 1994

The connection between continued fractions and orthogonality which is familiar for J-fractions and T-fractions is extended to what we call R-fractions of types I and II. These continued fractions are associated with recurrence relations that correspond to multipoint rational interpolants. A Favard type theorem is proved for each type. We then study explicit models which lead to biorthogonal rational functions.

1. INTRODUCTION

There is a close connection between the theory of orthogonal polynomials and continued fractions. This connection is clearest in the case of the classical moment problem where orthogonality is with respect to a positive measure whose Stieltjes transform is represented by a positive definite J-fraction (see [21, 31]). This connection is known to extend to the quasi-definite case where the measure and the J-fraction are no longer positive and positive definite, respectively [7, 12]. Another extension is by means of T-fractions and their connection with the trigonometric moment problem [20, 21, 29]. With both types of extensions one has a Favard type theorem establishing orthogonality with respect to a unique moment functional, given the three term recurrence relation satisfied by the polynomials under consideration. More importantly there is always a spectral measure(s) associated with these problems and by determining the regions where the associated continued fraction diverges one can locate the support of the corresponding spectral measure(s). Furthermore by finding the limits

* Research partially supported by NSF Grant DMS 9203659 and NSERC Grant A5384.

0021-9045/95 $12.00

to which the associated continued fraction converges one can compute, at least in theory, the spectral measure(s).

In this paper we shall examine two further natural extensions which, from a different point of view, are associated with multipoint rational interpolants or Padé approximants [10, 11, 25, 14, 28]. We call the new types of continued fractions "R-fractions." We establish a Favard type theorem for R-fractions. We also establish the existence of a natural Borel measure associated with R-fractions. In general we show the existence of rational biorthogonality with respect to this Borel measure. We demonstrate by means of examples how our general approach can yield explicit biorthogonal systems of rational functions and can lead to the evaluation of some types of beta integrals.

In Section 2 we consider what we call continued fractions of type R_I. Such continued fractions are associated with the polynomial recurrence relation

$$P_n(x) - (x - c_n) P_{n-1}(x) + \lambda_n (x - a_n) P_{n-2}(x) = 0. \tag{1.1}$$

In addition to proving a Favard type theorem for R_I-fractions in Section 2, we also establish a biorthogonality relation for rational functions. Section 2 contains, as an application of our results, a new evaluation of a q-beta integral of Ramanujan and an alternate derivation of a biorthogonality relation due to Pastro [26]. Pastro's original proof involves the aforementioned integral of Ramanujan. It is worth noting that we obtain both the evaluation of the q-beta integral and the biorthogonality relation from the same result on R_I-fractions. An additional result is the evaluation of the Herglotz transform of the Borel measure involved.

In Section 3 we consider continued fractions of type R_{II} which are associated with polynomial recurrence relation

$$P_n(x) - (x - c_n) P_{n-1}(x) + \lambda_n (x - a_n)(x - b_n) P_{n-2}(x) = 0. \tag{1.2}$$

In Section 3 we follow the same plan as in Section 2. We prove a representation theorem for moment functionals associated with (1.2) and study the convergence of the corresponding continued fraction. We also identify biorthogonal rational functions that arise naturally from the recurrence relation (1.2).

The letter R in our terminology is used to emphasize that the corresponding continued fractions are associated with rational interpolation and rational biorthogonality.

To illustrate our ideas we include seven examples where we have applied our general results to concrete situations. We have already mentioned that in Section 2, we rederive a result of Pastro [26]. In Section 3 we give two new systems of biorthogonal rational functions associated with continued fraction of type R_{II}. The first is based on the R_{II} recurrence relation with constant coefficients which corresponds to Chebyshev polynomials. The second uses ${}_2F_1$ contiguous relations and yields a weight function which is the integrand in the Cauchy beta integral. Section 4 contains two examples of biorthogonal ${}_4\phi_3$ rational functions based on solutions to the Askey–Wilson three term recurrence relation. The first rederives a system of biorthogonal rational functions due to Al-Salam and Ismail [1]. The second turns out to yield special cases of a biorthogonality related to q^{-1}-Hermite polynomials [18]. Section 5 contains two additional examples based on a modification of the Askey–Wilson recurrence. The first is new and unexpected. It provides an integral representation for a ${}_2\phi_1$ basic hypergeometric function and a special case of it leads to rational biorthogonality on $[-1, 1]$ based on the elementary integral

$$\frac{2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-x^2}\,dx}{(1-2\alpha x+\alpha^2)(1-2\delta x+\delta^2)}=\frac{1}{1-\alpha\delta}. \tag{1.3}$$

The second, which is originally due to Rahman [27], evaluates a q-beta integral of the type studied by Askey and Wilson and derives an associated system of rational functions biorthogonal on $[-1, 1]$. The biorthogonality is between a $\{{}_4\phi_3\}$ and $\{{}_6\phi_5\}$ pair. The methodology followed in the examples combines contiguous relations for hypergeometric or basic hypergeometric functions with Pincherle's theorem [21, 22]. This methodology proved fruitful in other types of continued fractions [17, 12, 16, 23, 32, 33].

We shall mostly follow the notation of Gasper and Rahman in [9]. The shifted factorials and multishifted factorials are

$$(a)_0 := 1, \qquad (a)_n := \prod_{j=1}^{n}(a+j-1), \qquad n>0, \tag{1.4}$$

and

$$(a_1, a_2, \cdots, a_k)_n := \prod_{j=1}^{k}(a_j)_n, \tag{1.5}$$

respectively, and a hypergeometric function is

$$\begin{aligned}{}_rF_s(a_1, a_2, \cdots, a_r; b_1, b_2, \cdots, b_s; z) &= {}_rF_s\left(\left.\begin{matrix}a_1, a_2, \cdots, a_r\\ b_1, b_2, \cdots, b_s\end{matrix}\right| z\right)\\ &:= \sum_{n=0}^{\infty}\frac{(a_1, a_2, \cdots, a_r)_n}{(b_1, b_2, \cdots, b_s)_n}\frac{z^n}{n!}.\end{aligned} \tag{1.6}$$

The q-shifted and multishifted factorials are similarly defined by

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{j=1}^{n} (1-aq^{j-1}), \qquad n>0, \text{ or } n=\infty, \tag{1.7}$$

$$(a_1, a_2, \cdots, a_k; q)_n := \prod_{j=1}^{k} (a_j; q)_n. \tag{1.8}$$

A basic hypergeometric series is

$$\begin{aligned} &{}_r\phi_s(a_1, a_2, \cdots, a_r; b_1, b_2, \cdots; q, z) \\ &= {}_r\phi_s\left(\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} \middle| q, z\right) \\ &:= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(b_1, b_2, \cdots, b_s; q)_n} \frac{[(-1)^n q^{n(n-1)/2}]^{1+s-r} z^n}{(q;q)_n}. \end{aligned} \tag{1.9}$$

Another useful notation is the shorthand notation for a very well-poised basic series. We shall use

$${}_8W_7(a; b, c, d, e, f; z) := {}_8\phi_7\left(\begin{matrix} a, \sqrt{aq}, -\sqrt{aq}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix} \middle| q, z\right). \tag{1.10}$$

The biorthogonality based on (1.3) which we alluded to earlier is

$$\frac{2}{\pi} \int_{-1}^{1} f_m(x; \delta, \alpha) f_n(x; \alpha, \delta) \sqrt{1-x^2}\, dx = \frac{(\alpha\delta)^n (q, q; q)_n}{(\alpha\delta, \alpha\delta; q)_n (1-\alpha\delta q^{2n})} \delta_{m,n}, \tag{1.11}$$

with $\max\{|\alpha|, |\delta|\} < 1$ and f_n is defined by

$$f_n(x; \alpha, \delta) := \frac{1}{1-2\alpha x + \alpha^2} \, {}_4\phi_3\left(\begin{matrix} q^{-n}, \alpha\delta q^n, \alpha e^{i\theta}, \alpha e^{-i\theta} \\ \alpha\delta, q\alpha e^{i\theta}, q\alpha e^{-i\theta} \end{matrix} \middle| q, q\right), \qquad x = \cos\theta. \tag{1.12}$$

The functions in (1.12) can be expressed in the simpler form

$$f_n(x; \alpha, \delta) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \frac{(\alpha\delta q^n; q)_k}{(\alpha\delta; q)_k} \frac{(-1)^k q^{k(1-k)/2}}{1-2\alpha q^k x + \alpha^2 q^{2k}}. \tag{1.13}$$

The expression $(q;q)_n/(q;q)_k (q;q)_{n-k}$ is the Gaussian binomial coefficient. The functions $\{f(x; \alpha, \delta)\}$ are biorthogonal with respect to the weight function of the Chebyshev polynomials of the second kind and may play an important role in a future theory of biorthogonal rational functions.

2. R_I-FRACTIONS

We begin this section with a Favard type theorem for the polynomial recurrence (2.1) below. We then outline how orthogonality may be realized in terms of the properties of what we will call an R_I type continued fraction. We also include an example to illustrate this approach.

We first establish a Favard type theorem which is Theorem 2.1. Consider a system of monic orthogonal polynomials generated by

$$P_n(x) = (x - c_n)\, P_{n-1}(x) - \lambda_n (x - a_n)\, P_{n-2}(x), \qquad P_{-1}(x) := 0, \qquad P_0(x) := 1, \tag{2.1}$$

where

$$\lambda_{n+1} \neq 0, \qquad P_n(a_{n+1}) \neq 0, \qquad n = 1, 2, \ldots.$$

The recurrence relation in (2.1) can be renormalized to yield the rational recurrence relation

$$(x - a_{n+1})\, R_n(x) - (x - c_n)\, R_{n-1}(x) + \lambda_n R_{n-2}(x) = 0 \tag{2.2}$$

with the same initial conditions, namely,

$$R_0(x) := 1, \qquad R_{-1}(x) := 0. \tag{2.3}$$

The renormalization is given explicitly by

$$R_n(x) := P_n(x) \Big/ \prod_{k=1}^{n} (x - a_{k+1}). \tag{2.4}$$

THEOREM 2.1. *Associated with the recurrence relation* (2.1) *there is a linear functional* $\mathscr{L}$ *defined on the span of* $\{x^k R_n(x)\}_{n,k=0}^{\infty}$ *mapping it into* $\mathscr{C}$, *and normalized by* $\mathscr{L}[1] = \lambda_1 \neq 0$, *such that the orthogonality relation*

$$\mathscr{L}[x^k R_n(x)] = 0, \qquad 0 \leqslant k < n, \tag{2.5}$$

holds. Furthermore the functional values $\mathscr{L}[x^n]$ *and* $\mathscr{L}[\prod_{k=1}^{n}(x - a_{k+1})^{-1}]$, $n = 1, 2, \ldots$, *are uniquely determined in terms of the sequences* $\{a_{n+1}, c_n, \lambda_n : n = 1, 2, \cdots\}$.

Proof. Define a linear functional $\mathscr{L}$ whose action on $R_n(x)$ and $x^{n-1}R_n(x)$ is given by

$$\mathscr{L}[1]:=\lambda_1, \qquad \mathscr{L}[R_n(x)]=\mathscr{L}[x^{n-1}R_n(x)]=0, \qquad \text{for } n\geqslant 1. \tag{2.6}$$

We first establish (2.5) for $1\leqslant k<n-1$. Clearly (2.6) yields

$$0=\mathscr{L}[R_1(x)]=\mathscr{L}[xR_2(x)]=\mathscr{L}[x^2R_3(x)]. \tag{2.7}$$

Apply $\mathscr{L}$ to (2.2) with $n=3$ to obtain $\mathscr{L}[xR_3(x)]=0$. Thus (2.5) holds for $1\leqslant n\leqslant 3$. When $n>3$, (2.5) follows from (2.2) and (2.6) by induction over n. To prove the latter part of the theorem observe that $\mathscr{L}[P_1(x)/(x-a_2)]=0$ implies $\mathscr{L}[1+P_1(a_2)/(x-a_2)]=0$. Using $\mathscr{L}[1]=\lambda_1$ we then obtain

$$\mathscr{L}[(x-a_2)^{-1}]=-\lambda_1/P_1(a_2).$$

Similarly, from $\mathscr{L}[P_2(x)/\{(x-a_2)(x-a_3)\}]=0$ we find

$$\mathscr{L}[1/\{(x-a_2)(x-a_3)\}]=\lambda_1(\lambda_2+a_3-a_2)/\{P_2(a_3)\,P_1(a_2)\}.$$

We next use $\mathscr{L}[x^kR_n(x)]$ for $n=1, 2$ and $k<n$ to obtain

$$\begin{aligned}\mathscr{L}[P_2(x)/(x-a_2)]&=\mathscr{L}[-\lambda_2+(x-c_2)\,P_1(x)/(x-a_2)]\\&=\mathscr{L}[x-c_1-\lambda_2]=0.\end{aligned}$$

Thus $\mathscr{L}[x]=(c_1+\lambda_2)\,\lambda_1$. We now continue by induction on n. Thus for each new n we use $\mathscr{L}[R_n(x)]=0$ and we obtain a value for $\mathscr{L}[1/\prod_{k=1}^{n}(1-a_{k+1})]$ while from $\mathscr{L}[x^{n-1}R_n(x)]=0$ and the recurrence relation (2.2) we evaluate $\mathscr{L}[x^{n-1}]$. This establishes our theorem.

COROLLARY 2.2. *We have*

$$\mathscr{L}[P_n(x)\,R_n(x)]=\mathscr{L}[x^nR_n(x)]=\lambda_1\lambda_2\ldots\lambda_{n+1}, \qquad n\geqslant 0. \tag{2.8}$$

Proof. Multiply (2.2) by x^{n-2}, apply $\mathscr{L}$, and then use the orthogonality relation (2.5). This yields the two term recurrence relation $\mathscr{L}[x^{n-1}R_{n-1}(x)]=\lambda_n\mathscr{L}[x^{n-2}R_{n-2}]$. Taking into account the initial condition $\mathscr{L}[1]=\lambda_1$ we establish (2.8) and the proof of the corollary is complete.

Note that when $a_n=0$ for $n\geqslant 2$ we have a Favard theorem associated with general T-fractions with all the moments $\mathscr{L}[x^n]$, $n=0, \pm 1, +2, \ldots$ uniquely determined [15, 20].

Our next result is a representation theorem for the functional $\mathscr{L}$ based on the properties of a continued fraction represented as an integral transform. Recall that the polynomials of the second kind associated with the recurrence relation (2.1) are given by

$$Q_n(x) - (x - c_n)\, Q_{n-1}(x) + \lambda_n (x - a_n)\, Q_{n-2}(x) = 0, \qquad n \geqslant 2, \quad (2.9)$$

with

$$Q_0(x) := 0, \qquad Q_1(x) := 1.$$

The ratio $Q_n(z)/P_n(z)$ is the nth convergent of the infinite continued fraction

$$R_I(z) = \frac{1}{z - c_1 -}\; \frac{\lambda_2(z - a_2)}{z - c_2} \;-\; \frac{\lambda_3(z - a_3)}{z - c_3} \;-\; \cdots \quad (2.10)$$

which we assume becomes a finite fraction when $z = a_k$, $k \geqslant 2$. This leads to the following definition.

Definition. A continued fraction of the type (2.10) will be referred to as an R_I-fraction provided that it terminates when $z = a_n$, $n \geqslant 2$.

We will henceforth assume that $R_I(z)$ converges to a function which vanishes at infinity and whose singularity structure is given by a finite number of branch cuts and a denumerable number of poles so that

$$R_I(z) := \int_\Gamma \frac{d\alpha(t)}{z - t}, \qquad z \in \mathbb{C} \backslash \mathcal{S}. \quad (2.11)$$

Here the measure $d\alpha(t)$ and the multiple contour Γ are taken in the following generalized sense. In order to accommodate a pole at z_k with multiplicity m_k and residue R_k we would include in the right side of (2.11) a term

$$\frac{1}{2\pi i} \int_{|t - z_k| > \varepsilon,\ |z - t| = \varepsilon} \frac{R_k\, dt}{(z - t)(t - z_k)^{m_k}},$$

with a suitably chosen positive ε. For a branch cut along a contour Γ_j with discontinuity $\alpha_j'(t)$ we would include a term

$$\frac{1}{2\pi i} \int_{\Gamma_j} \frac{\alpha_j'(t)\, dt}{z - t}, \qquad z \notin \Gamma_j.$$

We will also assume that the domain

$$\mathcal{S} = \left(\bigcup_{l=1}^{N} \Gamma_l \right) \cup \{ z_k : k = 1, 2, \ldots \}^{-}, \quad (2.12)$$

where $\overline{\{\cdot\}}$ the closure of the set $\{\cdot\}$. In other words, we assume that the continued fraction converges except at the singular points of the function that it represents.

From Pincherle's theorem [21] we then have the existence of a special minimal solution $X_n^{(\min)}(z)$ satisfying the same recurrence relation as $P_n(z)$, namely

$$X_n^{(\min)}(z) - (z - c_n)\, X_{n-1}^{(\min)}(z) + \lambda_n(z - a_n)\, X_{n-2}^{(\min)}(z) = 0, \qquad z \notin \mathscr{S} \qquad (2.13)$$

but with the minimality condition

$$\lim_{n \to \infty} X_n^{(\min)}(z)/P_n(z) = 0, \qquad z \notin \mathscr{S}.$$

Hence the additional representation

$$R_I(z) = \frac{X_0^{(\min)}(z)}{\lambda_1(z - a_1)\, X_{-1}^{(\min)}(z)} = \int_\Gamma \frac{d\alpha(t)}{z - t}, \qquad z \in \mathscr{C} \backslash \mathscr{S}. \qquad (2.14)$$

We will also assume the normalized asymptotics

$$R_I(z) = \frac{1}{z} + \sum_{n=2}^{\infty} d_n z^{-n}, \qquad |z| \to \infty, \qquad (2.15)$$

so that $\int_{-\infty}^{\infty} d\alpha(t) = 1$.

A further technical assumption will be that

$$\{a_n : 2 \leqslant n\} \cap \mathscr{S} = \varnothing. \qquad (2.16)$$

From the viewpoint of multiple point rational interpolants this means that the sequence of interpolation points associated with $R_I(z)$ is $\{\infty, a_2, \infty, a_3, \infty, ...\}$ with the a_n's distinct from the singular points of $\mathscr{S}$ [28].

Remark. Note that $R_I(z)$ in (2.10) does not depend on λ_1, a_1 while they occur in (2.14). This seeming dependence on λ_1, a_1 can be interpreted in two ways. First, the denominator $\lambda_1(z - a_1)\, X_{-1}^{(\min)}(z)$ is determined through the recurrence relation (2.13). That is,

$$\lambda_1(z - a_1)\, X_{-1}^{(\min)}(z) := (z - c_1)\, X_0^{(\min)}(z) - X_{-1}^{(\min)}(z).$$

Second, in explicit models where the sequences $\{\lambda_n\}$, $\{a_n\}$, and $\{X_n^{(\min)}(z)\}$ have an explicit analytic dependence on n, there are natural choices for λ_1, a_1, and $X_{-1}^{(\min)}(z)$. It often turns out that, for these natural values, $\lambda_1 = 0$. In these cases $(z - a_n)\, X_{n-2}^{(\min)}(z)$ is singular at $n = 1$ and the product

$\lambda_1(z-a_1)\,X_{-1}^{(\min)}(z)$ is indeterminant. For these cases it is convenient and correct to define $\lambda_1(z-a_1)\,X_{-1}^{(\min)}(z)$ through

$$\lambda_1(z-a_1)\,X_{-1}^{(\min)}(z) := \lim_{n\to 1} \lambda_n(z-a_n)\,X_{n-2}^{(\min)}(z).$$

This is also true for (2.18). A similar situation occurs for R_{II}-type fractions in (3.16) and (3.17).

We now state and prove a representation theorem.

THEOREM 2.3. *Consider the three term recurrence relation* (2.1) *and assume the representation* (2.11), *together with conditions* (2.12), (2.15), *and* (2.16). *Then the linear functional of Theorem* 2.1 *has the representation*

$$\mathscr{L}[f] = \int_{\Gamma} f(t)\,d\alpha(t). \tag{2.17}$$

Proof. Let $\lambda_1 = 1$. Then $\int_{\Gamma} d\alpha(t) = 1 = \lambda_1$. It remains to prove that $\int_{\Gamma} t^k R_n(t)\,d\alpha(t) = 0$, $0 \leqslant k < n$. This follows from the lemma below by taking the $z\to\infty$ asymptotics which yields for $0 \leqslant k < n$

$$z^{-1}\int_{\Gamma} t^k R_n(t)\,d\alpha(t) = z^{k-n-1}\lambda_1\lambda_2\cdots\lambda_{n+1}[1+O(1/z)],$$

where we have used the fact that $X_n^{(\min)}(z)/[\lambda_1 X_{-1}^{(\min)}(z)] \approx \prod_{j=2}^{n+1}\lambda_j$ as $z\to\infty$ which follows from (2.13)–(2.15).

LEMMA 2.4. *Consider the three term recurrence relation* (2.1) *and assume that the representation* (2.11), *together with the conditions* (2.12), (2.15), *and* (2.16) *hold. Then we have*

$$\frac{z^k X_n^{(\min)}(z)}{\lambda_1 X_{-1}^{(\min)}(z)\prod_{j=1}^{n+1}(z-a_j)} = \int_{\Gamma}\frac{t^k R_n(t)\,d\alpha(t)}{z-t}, \qquad 0 \leqslant k \leqslant n. \tag{2.18}$$

Proof. The singularities in (2.18) come from the zeros of $\lambda_1(z-a_1)$ $X_{-1}^{(\min)}(z)$ and the discontinuities in $X_n^{(\min)}(z)/\lambda_1(z-a_1)\,X_{-1}^{(\min)}(z)$ across the contours Γ_j, that is, the same singularities as in (2.14). The singularities which would seem to appear from the denominator factor $\prod_{j=1}^{n+1}(z-a_j)$ are, without loss of generality, canceled by corresponding zeros of $X_n^{(\min)}(z)$ (or alternatively poles of $\lambda_1 X_{-1}^{(\min)}(z)$ if one chooses a different normalization). In order to justify the right side of (2.18) it remains to compute the "weight" of the singularity. If $\lambda_1 X_{-1}^{(\min)}(z_j) = 0$, $X_0^{(\min)}(z_j) \neq 0$ then it follows from (2.12) and (2.1) that $X_n^{(\min)}(z_j) = P_n(z_j)\,X_0^{(\min)}(z_j)$ since each side of this equality satisfies the same recurrence relation and the same initial conditions. Thus for a pole at z_j of multiplicity m_j in (2.18) one has a residue

$$m_j!\, z_j^k\, P_n(z_j)\, X_0^{(\min)}(z_j) \Big/ \left[\lambda_1 \prod_{l=1}^{n+1} (z_j - a_l) \left(\frac{d^{m_j}}{dz^{m_j}} X_{-1}^{(\min)} \right)(z_j) \right]$$

as compared with the residue

$$X_0^{(\min)}(z_j)\, m_j! \Big/ \left[\lambda_1 (z_j - a_1) \left(\frac{d^{m_j}}{dz^{m_j}} X_{-1}^{(\min)} \right)(z_j) \right]$$

in (2.14). Hence we see that the pole singularities in (2.18) have a residue with an additional factor $z_j^k R_n(z_j)$ as required. The situation is similar for the absolutely continuous contribution. Thus if $\Delta(X_n^{(\min)}(z)/[\lambda_1(z-a_1)\, X_{-1}^{(\min)}(z)])$ is the discontinuity across a contour passing through the point z then $\Delta(X_n^{(\min)}(z)/\lambda_1(z-a_1)\, X_{-1}^{(\min)}(z)) = P_n(z)\, \Delta(X_0^{(\min)}(z)/\lambda_1(z-a_1)\, X_{-1}^{(\min)}(z))$, since each side of this equality again satisfies the same three term recurrence and the same initial conditions at $n=-1, 0$. The condition $0 \leqslant k \leqslant n$ is required so that (2.18) has at least $O(1/z)$ asymptotics and thus a zero at infinity.

We now illustrate the above theory with an example which is the q-analog of Jacobi–Laurent polynomials [15]. Although our example involves a special type of R_I-fraction, namely a general T-fraction, we consider it to be of intrinsic interest.

EXAMPLE 2.1 (${}_2\phi_1$ Functions). From the contiguous relations for ${}_2\phi_1$ hypergeometric functions in [17], or by expanding and equating coefficients of like powers of z, it can be verified that

$$X_{n+1}(z) - \left(z + q^{1/2} \frac{(1-bq^n)}{(1-aq^{n+1})} \right) X_n(z) + q^{1/2} z \frac{(1-q^n)(1-abq^n)}{(1-aq^{n+1})(1-aq^n)} X_{n-1}(z) = 0 \tag{2.19}$$

has solutions

$$X_n^{(1)}(z) := z^n \frac{(aq^{n+1}, bq^{n+1}; q)_\infty}{(q^{n+1}, abq^{n+1}; q)_\infty}\, {}_2\phi_1(1/a, q^{n+1}; bq^{n+1}; q, azq^{1/2}) \tag{2.20}$$

and

$$X_n^{(2)}(z) := q^{n/2} \frac{(aq^{n+2}, aq^{n+1}; q)_\infty}{(q^{n+1}, abq^{n+1}; q)_\infty}\, {}_2\phi_1(q/b, q^{n+1}; aq^{n+2}; q, bq^{1/2}/z) \tag{2.21}$$

and a polynomial solution

$$P_n(z) := \frac{q^{n/2}(b;q)_n}{(aq;q)_n}\, {}_2\phi_1(q^{-n}, aq; q^{1-n}/b; q, zq^{1/2}/b). \tag{2.22}$$

The large n asymptotics of these solutions is easily seen to be given by

$$X_n^{(1)}(z) \approx z^n (zq^{1/2};q)_\infty/(azq^{1/2};q)_\infty, \tag{2.23}$$

$$X_n^{(2)}(z) \approx q^{n/2}(q^{3/2}/z;q)_\infty/(bq^{1/2}/z;q)_\infty, \tag{2.24}$$

and

$$P_n(z) \approx \begin{cases} q^{n/2}\dfrac{(b, azq^{1/2};q)_\infty}{(aq, zq^{-1/2};q)_\infty}, & |z| < |q|^{1/2} \\ z^n \dfrac{(bq^{1/2}/z, qb;q)_\infty}{(q/b, q^{1/2}/z;q)_\infty}, & |z| > |q|^{1/2}. \end{cases} \tag{2.25}$$

Thus the minimal solution to the recurrence relation (2.19) is given by

$$X_n^{(\min)}(z) = \begin{cases} X_n^{(1)}(z), & |z| < |q|^{1/2} \\ X_n^{(2)}(z), & |z| > |q|^{1/2}. \end{cases} \tag{2.26}$$

Pincherle's theorem [21] then establishes the continued fraction representation

$$R_I(z) = \begin{cases} q^{-1/2}\dfrac{(1-aq)}{(1-b)}\, {}_2\phi_1(1/a, q; qb; q, zaq^{1/2}), & |z| < |q|^{1/2}, \\ z^{-1}\, {}_2\phi_1(q/b, q; aq^2; q, bq^{1/2}/z), & |z| > |q|^{1/2}, \end{cases} \tag{2.27}$$

where

$$R_I(z) := [z - c_1 - \mathbf{K}_{k=2}^{\infty}\{\lambda_k z/(z-c_k)\}]^{-1}, \tag{2.28}$$

with

$$c_n = -q^{1/2}\frac{(1-bq^{n-1})}{(1-aq^n)}, \qquad \lambda_n = q^{1/2}\frac{(1-q^{n-1})(1-abq^{n-1})}{(1-aq^n)(1-aq^{n-1})}. \tag{2.29}$$

Recall the Heine transformation [9, (III.1)]

$$ {}_2\phi_1(a, b; c; q, z) = \frac{(b, az;q)_\infty}{(c, z;q)_\infty}\, {}_2\phi_1(c/b, z; az; q, b). \tag{2.30}$$

From the above transformation we see that the ${}_2\phi_1$'s on the right side of (2.27) have singularities at $z=q^{-n-1/2}/a$ and $z=bq^{n+1/2}, n=0,1,\ldots$. However, because of the respective conditions $|z|<|q|^{1/2}$, $|z|>|q|^{1/2}$ the right side of (2.27) has no pole singularities if $|aq|<1$ and $|b|<1$. Thus when $|aq|<1$ and $|b|<1$ we also have the representation

$$R_I(z)=\int_{|t|=|q|^{1/2}} \frac{\alpha'(t)\,dt}{z-t}, \qquad |z|\neq q^{1/2}, \tag{2.31}$$

with α' given by

$$\alpha'(t):=\lim_{\varepsilon\to 0^+,\, n\to 0} \frac{1}{2\pi i\lambda_{n+1} t}\left(\frac{X_n^{(\min)}(t_+)}{X_{n-1}^{(\min)}(t_+)}-\frac{X_n^{(\min)}(t_-)}{X_{n-1}^{(\min)}(t_-)}\right), \qquad t=|q|^{1/2}e^{i\theta},$$

$$t_{\pm}:=(|q|^{1/2}\pm\varepsilon)\,e^{i\theta}, \qquad 0\leqslant\theta\leqslant 2\pi. \tag{2.32}$$

A calculation using (2.26),(2.30), and the q-Vandermonde nonterminating sum [9, II.23] yields

$$\alpha'(t)=\frac{i}{2\pi q^{1/2}}\,\frac{(q^{1/2}t, q^{1/2}/t, q, abq; q)_\infty}{(aq^{1/2}t, bq^{1/2}/t, aq^2, b; q)_\infty}. \tag{2.33}$$

From our general theory associated with Theorem 2.3 or the general theory of T-fractions [20] we may then state the following results:

$$\int_{|t|=|q|^{1/2}} t^{-k}P_m(t)\,\alpha'(t)\,dt=0, \qquad 0<k\leqslant m, \tag{2.34}$$

$$\int_{|t|=|q|^{1/2}} P_m(t)\,\alpha'(t)\,dt=\frac{q^{m/2}(q, abq; q)_m}{(aq, aq^2; q)_m}, \tag{2.35}$$

and

$$-\int_{|t|=|q|^{1/2}} t^{-m-1}\,P_m(t)\,\alpha'(t)\,dt=\frac{(1-aq)\,q^{-1/2}\,(abq, q; q)_m}{(1-b)\,(aq, bq; q)_m}. \tag{2.36}$$

We now derive a rational biorthogonality using (2.34) and (2.36). As a first step we shift the contour of integration in (2.34) and (2.36) to the unit circle $|t|=1$ in order to obtain

$$\int_0^{2\pi} t^{-k}\,P_m(t) f(a,b,t)\,d\theta=\frac{(q, abq; q)_m}{(aq, bq; q)_m}\,\delta_{k,m},$$

$$0\leqslant k\leqslant m, \qquad t=e^{i\theta} \tag{2.37}$$

with $f(a, b, t)$ given by

$$f(a, b, t)=\frac{1}{2\pi}\frac{(q^{1/2}t, q^{1/2}/t, q, qab; q)_\infty}{(aq^{1/2}t, bq^{1/2}/t, qa, qb; q)_\infty}.$$

The condition for shifting the contour is that no poles of $\alpha'(t)$ are crossed as one goes from $|t|=|q|^{1/2}$ to $|t|=1$. Thus one needs $|q^{-n+1/2}|>|a|$ and $|bq^{n-1}|<1$, $n=1, 2, ...$, i.e., $|a|<|q|^{-1/2}$ and $|b|<1$. The latter two conditions are satisfied since we assumed that $|a|<|q|^{-1}$ and $|b|<1$.

We now take the complex conjugate of (2.37) followed by the replacements $(q, a, b)\rightarrow(\bar{q}, \bar{b}, \bar{a})$ to obtain

$$\int_0^{2\pi} t^k Q_m(1/t) f(a, b, t)\, d\theta=\frac{(q, abq; q)_m}{(aq, bq; q)_m}\delta_{k,m}, \qquad 0\leqslant k\leqslant m, \quad (2.38)$$

where $Q_m(z)$ is the polynomial

$$Q_m(z)=\frac{q^{m/2}(a; q)_m}{(bq; q)_m}\,{}_2\phi_1(q^{-m}, bq; q^{1-m}/a; q, zq^{1/2}/a). \qquad (2.39)$$

From (2.37) and (2.38) we finally obtain the biorthogonality relation

$$\int_0^{2\pi} P_m(t)\, Q_n(1/t) f(a, b, t)\, d\theta=\frac{(q, abq; q)_m}{(aq, bq; q)_m}\delta_{m,n}, \qquad |a|<1, \quad |b|<1. \qquad (2.40)$$

This biorthogonality had been previously obtained by Pastro [26] using other methods. In [1] the biorthogonality (2.40) was used to derive a biorthogonality for ${}_4\phi_3$ rational functions. Here we have shown that the biorthogonality (2.40) is a byproduct of the theory of R_I fractions and we have derived a more general transform given by (2.18); namely, if $|aq|<1$, $|b|<1$, $0\leqslant k\leqslant n$, then

$$\frac{i}{2\pi}\int_{|t|=|q|^{1/2}}\frac{t^{k-n}P_n(t)}{z-t}\frac{(q^{1/2}t, q^{1/2}/t; q)_\infty}{(aq^{1/2}t, bq^{1/2}/t; q)_\infty}\,dt$$

$$=\begin{cases} z^k\dfrac{(aq^{n+1}, bq^{n+1}; q)_\infty}{(q^{n+1}, abq^{n+1}; q)_\infty}\,{}_2\phi_1(1/a, q^{n+1}; bq^{n+1}; q, azq^{1/2}), & |z|<|q|^{1/2}, \\ q^{(n+1)/2-k-n-1}\dfrac{(aq^{n+1}, aq^{n+2}, b; q)_\infty}{(q^{n+1}, abq^{n+1}, aq; q)_\infty} & \\ \quad\times\,{}_2\phi_1(q/b, q^{n+1}; aq^{n+2}; q, bq^{1/2}/z), & |z|>|q|^{1/2}, \end{cases} \qquad (2.41)$$

where $P_n(z)$ is as in (2.22).

The case $z=k=n=0$ in (2.41) is equivalent to the original q-beta integral of Ramanujan with which Pastro started [26].

3. R_{II}-Fractions

The format of Section 2 is repeated here with first a Favard type theorem and then its realization in terms of the properties of an R_{II} type continued fraction. We include an example of a new system of biorthogonal rational functions. In Section 4 we cast the recent results of [1, 18] in the language of R_{II} fractions and use the theory outlined in this work to give new derivations of these results. We also discover a new set of biorthogonal rational functions in Section 5.

Consider the system of polynomials $\{P_n(x)\}$ satisfying the recurrence relation

$$P_n(x)-(x-c_n)\,P_{n-1}(x)+\lambda_n(x-a_n)(x-b_n)\,P_{n-2}(x)=0, \qquad n\geqslant 1, \tag{3.1}$$

and the additional assumptions

$$P_{-1}(x)=0, \qquad P_0(x)=1, \qquad \lambda_{n+1}\neq 0, \qquad P_n(a_{n+1})\neq 0,$$
$$P_n(b_{n+1})\neq 0, \qquad n>0.$$

The recurrence relation (3.1) can be renormalized to yield the rational function recurrence relation

$$(x-a_{n+1})(x-b_{n+1})\,S_n(x)-(x-c_n)\,S_{n-1}(x)+\lambda_n S_{n-2}(x)=0, \qquad n>1, \tag{3.2}$$

with

$$S_n(x)=P_n(x)\Big/\prod_{k=1}^{n}\,[(x-a_{k+1})(x-b_{k+1}). \tag{3.3}$$

We now come to an analog of Favard's theorem.

THEOREM 3.5. *Given the recursion* (3.1) *there is a linear functional* $\mathscr{L}$ *defined on the span of the rational functions* $\{x^kS_n(x):0\leqslant k\leqslant n<\infty\}$, *mapping it into* $\mathscr{C}$ *and normalized by* $\mathscr{L}[1]=N_0$, $\mathscr{L}[xS_1(x)]=N_1$ *such that the orthogonality relation* $\mathscr{L}[x^kS_n(x)]=0$, $0\leqslant k<n$, *holds. Furthermore, the values of* $\mathscr{L}[\prod_{j=1}^{n}(x-a_{j+1})^{-1}\prod_{k=1}^{m}(x-b_{j+1})^{-1}]$ *for* $m, n=0, 1, \ldots,$ *are uniquely determined.*

Proof. Define a linear functional $\mathscr{L}$ by requiring it to satisfy $\mathscr{L}[1]=N_0$, $\mathscr{L}[xS_1(x)]=N_1$, $\mathscr{L}[S_1(x)]=0$, and $\mathscr{L}[S_n(x)]=\mathscr{L}[xS_n(x)]=0$, $n\geqslant 2$. The recurrence relation (3.2) then yields $\mathscr{L}[x^kS_n(x)]=0$, $1<k<n$. From $\mathscr{L}[xS_1(x)]=N_1$ and $\mathscr{L}[S_1(x)]=0$ we obtain $\mathscr{L}[(x-c_1)/(x-a_2)]=\mathscr{L}[(x-c_1)/(x-b_2)]=N_1$ and, hence,

$$\mathscr{L}[1+(a_2-c_1)/(x-a_2)]=\mathscr{L}[1+(b_2-c_1)/(x-b_2)]=N_1.$$

With $\mathscr{L}[1]=N_0$ this implies $\mathscr{L}[(x-a_2)^{-1}]=(N_1-N_0)/(a_2-c_1)$ and $\mathscr{L}[(x-b_2)^{-1}]=(N_1-N_0)/(b_2-c_1)$. If $a_2\neq b_2$ then a partial fraction decomposition yields

$$\begin{aligned}\mathscr{L}[(x-a_2)^{-1}(x-b_2)^{-1}]&=(a_2-b_2)^{-1}\,\mathscr{L}[(x-a_2)^{-1}-(x-b_2)^{-1}]\\&=(N_0-N_1)/[(a_2-c_1)(b_2-c_1)].\end{aligned}$$

On the other hand, if $a_2=b_2$ then $\mathscr{L}[S_1(x)]=0$ yields $\mathscr{L}[(x-a_2)^{-1}+(a_2-c_1)(x-a_2)^{-2}]=0$, which implies $\mathscr{L}[(x-a_2)^{-2}]=(a_2-c_1)^{-2}(N_0-N_1)$. We continue this argument by induction on n using $\mathscr{L}[x^kS_n(x)]=0$, $0\leqslant k<n$, $n\geqslant 2$. Thus for each new n, $n\geqslant 2$ we use $\mathscr{L}[S_n(x)]=\mathscr{L}[xS_n(x)]=0$ to evaluate $\mathscr{L}[\prod_{j=1}^{N}(x-a_{j+1})^{-1}\prod_{k=1}^{M}(x-b_{j+1})^{-1}]$, $N=n$, $M<n$, $N<n$, $M=n$, and $N=M=n$.

The next result gives a recursive definition of $\mathscr{L}[x^nS_n(x)]$.

COROLLARY 3.6. *Set $\mathscr{L}[x^nS_n(x)]=N_n$. Then the N_n's satisfy the three term recurrence relation*

$$N_n-N_{n-1}+\lambda_nN_{n-2}=0,\qquad n\geqslant 2. \tag{3.4}$$

Proof. Multiply the recursion (3.2) by x^{n-2}, apply $\mathscr{L}$ and make use of the orthogonality relation $\mathscr{L}[x^kS_n(x)]=0$, $0\leqslant k<n$. The result is (3.4).

It is because of (3.4) that we need two initial normalizations N_0 and N_1. In Example 3.2, which comes later in this section, we will see that $N_n=\kappa_1\kappa_2\cdots\kappa_{n+1}$ with $\kappa_1=1/(1-\kappa_2)$. Now $\kappa_j=\lambda_j/(1-\kappa_{j+1})$, $j\geqslant 2$, follows from (3.4). Since the continued fraction $\mathbf{K}_{n=2}^{\infty}(\lambda_n/1)$ of the example converges we find that (3.4) can be realized in practice by having

$$\kappa_j=\mathbf{K}_{n=j}^{\infty}\left(\frac{\lambda_n}{1}\right),\qquad j>1,\qquad \kappa_1=(1-\kappa_2)^{-1}, \tag{3.5}$$

and

$$N_n=\kappa_1\kappa_2\cdots\kappa_{n+1}. \tag{3.6}$$

We now come to a representation theorem for R_{II}-fractions. The polynomials of the second kind associated with the recursion (3.1) are generated by

$$Q_n(x)-(x-c_n)\,Q_{n-1}(x)+\lambda_n(x-a_n)(x-b_n)\,Q_{n-2}(x)=0, \qquad n>1,$$
$$Q_0(x):=0, \qquad Q_1(x):=1. \tag{3.7}$$

The ratio $Q_n(z)/P_n(z)$ is the nth convergent (approximant) of the continued fraction

$$R_{II}(z)=\frac{1}{z-c_1-}\;\frac{\lambda_2(z-a_2)(z-b_2)}{z-c_2}\;-\;\frac{\lambda_3(z-a_3)(z-b_3)}{z-c_3}\;-\;\cdots, \tag{3.8}$$

which we assume becomes a finite fraction for $z=a_k$ or $z=b_k$ for any $k>1$. This leads to the following definition.

Definition. A continued fraction of the type (3.8) will be referred to as an R_{II}-fraction provided that it terminates in the cases $z=a_n$ and $z=b_n$, $n\geqslant 1$.

We will henceforth assume that $R_{II}(z)$ converges to a function which vanishes at infinity and whose singularity structure is given by at most a finite number of branch cuts $\{\Gamma_j\}_1^N$ and a denumerable number of poles $\{z_j\}_1^\infty$ so that, in the generalized sense previously explained in Section 2, the following representation holds:

$$R_{II}(z)=\int_\Gamma \frac{d\alpha(t)}{z-t}, \qquad z\in\mathbb{C}\setminus\mathscr{D}. \tag{3.9}$$

Also, as before, we assume that $\mathscr{D}$ is minimal so that

$$\mathscr{D}=\left(\bigcup_{j=1}^N \Gamma_j\right)\cup\{z_k\}^- \tag{3.10}$$

and that the set of interpolation points $\{a_n, b_n\}_{n=2}^\infty$ (see [28]) is disjoint from the set of singular points $\mathscr{D}$; that is,

$$\{a_n, b_n\}_{n=2}^\infty\cap\mathscr{D}=\varnothing. \tag{3.11}$$

We will also assume the large z asymptotics

$$R_{II}(z)\approx\kappa_1/z=z^{-1}\int_\Gamma d\alpha(t), \tag{3.12}$$

with

$$\kappa_1=\frac{1}{1-}\;\frac{\lambda_2}{1-}\;\frac{\lambda_3}{1-}\;\cdots \tag{3.13}$$

a convergent continued fraction.

THEOREM 3.7. *Consider the three term recurrence relation* (3.1) *and assume that the representation* (3.9) *and the conditions* (3.10)–(3.13) *hold. Then the linear functional* $\mathscr{L}$ *of Theorem* 3.1 *has the integral representation*

$$\mathscr{L}[f] = \int_{\Gamma} f(t)\, d\alpha(t). \tag{3.14}$$

Furthermore, the normalization constants of Corollary 3.2 *are realized by* (3.5) *and* (3.6).

Proof. We set $N_0 = \kappa_1$. It remains to show that $N_1 = \kappa_1 - 1$ and that

$$\int_{\Gamma} t^k S_n(t)\, d\alpha(t) = 0, \qquad 0 \leqslant k < n. \tag{3.15}$$

This will follow from the large z asymptotics in (3.16) of the lemma below.

LEMMA 3.8. *Under the assumptions of Theorem* 3.3 *we have the integral representation*

$$\frac{z^k X_n^{(\min)}(z)}{\lambda_1 \prod_{j=1}^{n+1} [(z-a_j)(z-b_j)]\, X_{-1}^{(\min)}(z)} = \int_{\Gamma} \frac{t^k S_n(t)}{z-t}\, d\alpha(t), \quad 0 \leqslant k \leqslant n, \quad z \in \mathbb{C} \backslash \mathscr{D}. \tag{3.16}$$

where

$$\lambda_1 (z-a_1)(z-b_1)\, X_{-1}^{(\min)}(z) = (z-c_1)\, X_0^{(\min)}(z) - X_1^{(\min)}(z)$$

(*see the remark before Theorem* 2.3). *In* (3.16) $X_n^{(\min)}(z)$ *denotes the minimal solution to the recurrence relation* (3.1).

Proof. Here again we invoke Pincherle theorem and establish the representation

$$\frac{X_0^{(\min)}(z)}{\lambda_1 (z-a_1)(z-b_1)\, X_{-1}^{(\min)}(z)} = \int_{\Gamma} \frac{d\alpha(t)}{z-t}, \qquad z \in \mathbb{C} \backslash \mathscr{D}. \tag{3.17}$$

The asymptotic relationship (3.12) gives

$$\frac{X_0^{(\min)}(z)}{\lambda_1 X_{-1}^{(\min)}(z)} \approx \kappa_1 z. \tag{3.18}$$

The three term recurrence relation then implies

$$X_1^{(\min)}(z)/X_0^{(\min)}(z) \approx (1-1/\kappa_1)\, z = \kappa_2 z,$$

say, with $\kappa_2 := (1-1/\kappa_1)$ and

$$X_n^{(\min)}(z)/X_{n-1}^{(\min)}(z) \approx z(1-\lambda_n/\kappa_n) = \kappa_{n+1} z, \qquad n \geqslant 2.$$

This establishes the large z asymptotics on the left of (3.16) as

$$\frac{z^k X_n^{(\min)}(z)}{\lambda_1 \left[\prod_{j=1}^{n+1}(z-a_j)(z-b_j)\right] X_{-1}^{(\min)}(z)} \approx z^{k-n-1} N_n, \tag{3.19}$$

with $N_n = \kappa_1 \kappa_2 \cdots \kappa_{n+1}$ and the κ's are given by (3.5). To establish the equality (3.16) we follow the same procedure as in Lemma 2.4. This means that the singularities of (3.16) and (3.17) are the same but (3.16) has the additional weight factor $t^k S_n(t)$, $k \leqslant n$.

Note that the large z asymptotics of (3.16) yields

$$\int_{\Gamma} t^k S_n(t)\, d\alpha(t) \approx z^{k-n} N_n, \qquad 0 \leqslant k \leqslant n.$$

The choice $k=n=1$ yields $N_1 = \kappa_1 \kappa_2 = \kappa_1 - 1$ but the remaining choices give the orthogonality relations $\int_\Gamma t^k S_n(t)\, d\alpha(t) = 0$, $0 \leqslant k < n$, as required by Theorem 3.3.

EXAMPLE 3.1 (Chebyshev polynomials of type R_{II}). This case with constant coefficients is given in [24] with some misprints which are corrected here. Consider the recurrence relation (3.1) of the form

$$X_{n+1}(z) - (z+\sqrt{ab})\, X_n(z) + \tfrac{1}{4}(z-a)(z-b)\, X_{n-1}(z) = 0, \qquad a, b > 0. \tag{3.20}$$

This difference equation has solutions

$$X_n^{\pm}(z) = \left(\frac{(\sqrt{z} \pm \sqrt{a})(\sqrt{z} \pm \sqrt{b})}{2}\right)^n. \tag{3.21}$$

For $z \notin (-\infty, 0]$ the minimal solution is therefore

$$X_n^{(\min)}(z) = \left(\frac{(\sqrt{z} - \sqrt{a})(\sqrt{z} - \sqrt{b})}{2}\right)^n \tag{3.22}$$

with the square root branch chosen so that

$$|z+\sqrt{ab} - (\sqrt{a}+\sqrt{b})\sqrt{z}| < |z+\sqrt{ab} + (\sqrt{a}+\sqrt{b})\sqrt{z}|.$$

From Pincherle's theorem we then have

$$\frac{1}{z+\sqrt{ab}-}\,\frac{(z-a)(z-b)/4}{z+\sqrt{ab}}\;-\;\frac{(z-a)(z-b)/4}{z+\sqrt{ab}}\;-\;\cdots$$
$$=\frac{2}{(\sqrt{z}+\sqrt{a})(\sqrt{z}+\sqrt{b})}$$
$$=\frac{2}{\pi}\int_{-\infty}^{0}\frac{(\sqrt{a}+\sqrt{b})\sqrt{-x}\,dx}{(a-x)(b-x)(z-x)}. \tag{3.23}$$

Corresponding to the more general formula (3.16) we also have for $0 \leqslant m \leqslant n$,

$$\frac{P_m(z)\,X_n^{(\min)}(z)(\sqrt{z}-\sqrt{a})(\sqrt{z}-\sqrt{b})}{(z-a)^{n+1}(z-b)^{n+1}}$$
$$=\frac{1}{\pi}\int_{-\infty}^{0}\frac{(\sqrt{a}+\sqrt{b})\sqrt{-x}\,P_m(x)\,P_n(x)\,dx}{(a-x)^{n+1}(b-x)^{n+1}(z-x)}, \tag{3.24}$$

where

$$P_n(z)=\frac{[(\sqrt{z}+\sqrt{a})(\sqrt{z}+\sqrt{b})]^{n+1}-[(\sqrt{z}-\sqrt{a})(\sqrt{z}-\sqrt{b})]^{n+1}}{2^{n+1}(\sqrt{a}+\sqrt{b})\sqrt{z}} \tag{3.25}$$

is the polynomial solution (no longer monic) with initial values $P_{-1}=0$, $P_0=1$.

From the large z asymptotics of (3.24) and (3.25) we easily obtain the orthogonality for $m \leqslant n$

$$\frac{2}{\pi}\int_{-\infty}^{0}\frac{(\sqrt{a}+\sqrt{b})\,P_m(x)\,P_n(x)\sqrt{-x}\,dx}{(a-x)^{n+1}(b-x)^{n+1}}=2^{-2n+1}(n+1)\,\delta_{m,n}. \tag{3.26}$$

If we now introduce the rational functions

$$R_{2n}(x):=\frac{P_{2n}(x)}{(a-x)^n(b-x)^n},\qquad R_{2n+1}(x):=\frac{P_{2n+1}(x)}{(a-x)^n(b-x)^{n+1}},\qquad n\geqslant 0, \tag{3.27}$$

then (3.24) (with $m=n$ and $z=a$) and (3.26) (with $n\neq m$) translate into the rational orthogonality

$$\frac{2}{\pi}\int_{-\infty}^{0}\frac{(\sqrt{a}+\sqrt{b})\sqrt{-x}\,R_m(x)\,R_n(x)\,dx}{(a-x)^2(b-x)}=\frac{2^{-2n}a^{-1/2}}{(\sqrt{a}+\sqrt{b})}\,\delta_{m,n}. \tag{3.28}$$

EXAMPLE 3.2 ($_2F_1$ Functions). A contiguous relation for a hypergeometric function of type $_2F_1$ in [8, (45), p. 104] tells us that the recurrence relation

$$X_n(z)=\left(z-\frac{n+a-1}{2n+a-1-b}\right)X_{n-1}(z) - \frac{z(z-1)(n-1)(n+a-1-b)}{(2n+a-1-b)(2n+a-3-b)}X_{n-2}(z)=0 \tag{3.29}$$

has solutions

$$X_n^{(1)}(z):=\left(\frac{z}{2}\right)^n \frac{\Gamma(n+1)\,\Gamma(n+1+a-b)}{\Gamma(n+a+1)\,\Gamma(n+(1+a-b)/2)}\,{}_2F_1(a,b;n+1+a;z), \tag{3.30}$$

and

$$X_n^{(2)}(z):=\left(\frac{z-1}{2}\right)^n \frac{\Gamma(n+1)\,\Gamma(n+1+a-b)}{\Gamma(n+2-b)\,\Gamma(n+(1+a-b)/2)} \times {}_2F_1(1-a,1-b;n+2-b;1-z) \tag{3.31}$$

and a polynomial solution

$$X_n^{(3)}(z)=P_n(z;a,b):=\frac{2^{-n}(1-b)_n}{((1+a-b)/2)_n}\,{}_2F_1(-n,b-a-n;b-n;1-z). \tag{3.32}$$

The asymptotic behavior of $X_n^{(1)}(z)$ and $X_n^{(2)}(z)$ as $n\to\infty$ can be easily calculated and it yields

$$X_n^{(1)}(z)\approx(z/2)^n n^{-(a+b-1)/2}, \qquad |z|<1, \tag{3.33}$$

$$X_n^{(2)}(z)\approx\left(\frac{z-1}{2}\right)^n n^{(a+b-1)/2}, \qquad |z-1|<1. \tag{3.34}$$

Thus the minimal solution of (3.29) is given by

$$X_n^{(\min)}(z)=\begin{cases} X_n^{(1)}(z), & \Re z<1/2 \\ X_n^{(2)}(z), & \Re z>1/2. \end{cases} \tag{3.35}$$

Exploiting Pincherle's theorem we obtain

$$R_{II}(z)=\begin{cases} -\dfrac{(1+a-b)(1-z)^{b-1}}{a}\,{}_2F_1(a,b;1+a;z), & \mathcal{R}z<1/2,\\ -\dfrac{(1+a-b)}{b-1}z^{-a}\,{}_2F_1(1-a,1-b;2-b;1-z), & \mathcal{R}z>1/2, \end{cases} \tag{3.36}$$

where

$$R_{II}(z)=[z-c_1+\mathbf{K}_{n=2}^{\infty}\{z(z-1)\,\lambda_n/(z-c_n)\}]^{-1},$$
$$c_n:=\frac{(n+a-1)}{(2n+a-b-1)},\qquad \lambda_n:=\frac{(n-1)(n+a-1-b)}{(2n+a-1-b)(2n+a-3-b)}. \tag{3.37}$$

We now have an explicit example of an R_{II} type continued fraction which converges, except possibly when $\mathcal{R}z=\frac{1}{2}$. To calculate the absolutely continuous measure which is now supported on $\mathcal{R}z=\frac{1}{2}$ we use [8, 2.9(33), p. 107],

$$\frac{(1+a-b)}{a}(1-z)^{b-1}{}_2F_1(a,b;1+a;z)$$
$$-\frac{(1+a-b)}{(b-1)}z^{-a}{}_2F_1(1-a,1-b;2-b;1-z)$$
$$=z^{-a}(1-z)^{b-1}\frac{(1+a-b)\,\Gamma(a)\,\Gamma(1-b)}{\Gamma(1+a-b)}. \tag{3.38}$$

Thus we have established

$$R_{II}(z)=\frac{(1+a-b)\,\Gamma(a)\,\Gamma(1-b)}{2\pi i\,\Gamma(1+a-b)}\int_{1/2-i\infty}^{1/2+i\infty}\frac{t^{-a}(1-t)^{b-1}}{z-t}\,dt,\qquad \mathcal{R}z\neq 1/2, \tag{3.39}$$

with the correct $O(1/z)$ asymptotics of the present section when $\mathcal{R}(a-b)>0$ which is [8, 2.1.4(17), p. 63]

$$R_{II}(z)\approx\left(\frac{a-b+1}{a-b}\right)z^{-1},\qquad \mathcal{R}(a-b)>0. \tag{3.40}$$

This justifies the assumption (3.12) since we may rewrite the λ_n's in the form

$$\lambda_n=\frac{1}{4}+\frac{1-(a-b)^2}{4(n+(a-b-3)/2))(n+(a-b-1)/2)} \tag{3.41}$$

and a result in [30] yields

$$\kappa_1 = \frac{1}{1-}\frac{\lambda_2}{1-}\frac{\lambda_3}{1-}\cdots = \frac{a-b+1}{a-b}, \qquad \mathscr{R}(a-b)>0. \tag{3.42}$$

We now derive a biorthogonality relation for certain rational functions using the orthogonality relation (3.15) with the special case $k \leqslant n$, $z=0$, of the integral formula (3.16). We set

$$U_n(x; a, b) := \frac{P_n(x; a, b)}{(x-1)^n}, \qquad V_n(x; a.b) := U_n(1-x; -b, -a), \tag{3.43}$$

$$g(x; a, b) := x^{-a-1}(1-x)^{b-1}/2\pi i. \tag{3.44}$$

The orthogonality relation (3.15) now takes the form

$$\int_{1/2-i\infty}^{1/2+i\infty} t^{-m} U_n(t; a, b)\, g(t; a, b)\, dt = 0, \qquad 0 \leqslant m < n. \tag{3.45}$$

Taking the complex conjugate of (3.45), replacing a by $-\bar{b}$ and b by $-\bar{a}$, taking into account that $\bar{t} = 1-t$ and using (3.16) (with z^k replaced by $P_k(1-z; -b, -a)$, $k=n$, and $z=0$) we see that

$$\int_{1/2-i\infty}^{1/2+i\infty} U_n(t; a, b)\, V_m(t; a, b)\, g(t; a, b)\, dt$$

$$= \frac{\Gamma(1+a-b)\, n!(1+a-b)_n}{\Gamma(1+a)\,\Gamma(1-b)\, 2^{2n}[((a-b+1)/2)_n]^2}\,\delta_{m,n}, \tag{3.46}$$

provided that $a, b \neq 0$, $a \neq -1$, $b \neq 1$, $\mathscr{R}(a-b)>0$.

The biorthogonal rational functions in this example differ from the ones considered by Askey in [2]. Both systems of rational functions are biorthogonal with respect to a Cauchy beta integral weight function.

4. The ${}_4\phi_3$ Functions

In this section we provide two examples of biorthogonal rational functions which have explicit representation as ${}_4\phi_3$'s. The three term recurrence relations and the continued fractions associated with these systems come from contiguous relations for ${}_4\phi_3$ and ${}_8\phi_7$ functions. The ${}_4\phi_3$ contiguous relation is the same one used by Askey and Wilson [5] and that led to the Askey–Wilson polynomials. The difference between our examples and the Askey–Wilson polynomials is a reparametrization which converts the three term recurrence to one of type R_{II}.

The monic Askey–Wilson polynomials are defined by [5]

$$\begin{aligned} &P_n(x;\alpha,\beta,\gamma,\delta|q) \\ &:=\frac{(\alpha\beta,\alpha\gamma,\alpha\delta,\alpha\beta\gamma\delta/q;q)_n}{(2\alpha)^n(\alpha\beta\gamma\delta/q;q)_{2n}}\,{}_4\phi_3\left(\begin{matrix}q^{-n},\alpha\beta\gamma\delta q^{n-1},\alpha u,\alpha/u\\ \alpha\beta,\alpha\gamma,\alpha\delta\end{matrix}\,\middle|\,q,q\right), \end{aligned} \tag{4.1}$$

where

$$x:=\tfrac{1}{2}(u+1/u). \tag{4.2}$$

The P_n's satisfy the three term recurrence relation

$$\begin{aligned} &P_n(x)-(x-a_{n-1})P_{n-1}(x)+A_{n-2}B_{n-1}P_{n-2}(x)=0,\\ &a_n=\tfrac{1}{2}(\alpha+1/\alpha)-A_n-B_n,\\ &A_n:=\frac{(1-\alpha\beta\gamma\delta q^{n-1})(1-\alpha\beta q^n)(1-\alpha\gamma q^n)(1-\alpha\delta q^n)}{2\alpha(1-\alpha\beta\gamma\delta q^{2n-1})(1-\alpha\beta\gamma\delta q^{2n})},\\ &B_n=\frac{\alpha(1-q^n)(1-\beta\gamma q^{n-1})(1-\beta\delta q^{n-1})(1-\gamma\delta q^{n-1})}{2(1-\alpha\beta\gamma\delta q^{2n-2})(1-\alpha\beta\gamma\delta q^{2n-1})}. \end{aligned} \tag{4.3}$$

In [13] the minimal solution to the recurrence relation (4.3) for $n>0$ has been shown to be

$$Y_n^{(\min)}(x)=\begin{cases}F_n(u), & |u|>1,\\ F_n(1/u), & |u|<1,\end{cases} \tag{4.4}$$

and the F_n's are the functions

$$\begin{aligned} F_n(u):=&\frac{(2u)^{-n}(\alpha\beta\gamma\delta q^{2n-1},\alpha q^{n+1}/u,\beta q^{n+1}/u,\gamma q^{n+1}/u,\delta q^{n+1}/u;q)_\infty}{(q^{n+1},q^{n+2}/u^2,\alpha\beta q^n,\alpha\gamma q^n,\alpha\delta q^n,\beta\gamma q^n,\beta\delta q^n,\gamma\delta q^n;q)_\infty}\\ &\times{}_8W_7\left(q^{n+1}/u^2;q^{n+1},\frac{q}{\alpha u},\frac{q}{\beta u},\frac{q}{\gamma u},\frac{q}{\delta u};\alpha\beta\gamma\delta q^{n-1}\right). \end{aligned} \tag{4.5}$$

From (4.4) one can calculate, using Pincherle's theorem, the Askey–Wilson weight function

$$w(x):=\frac{1}{2\pi\sqrt{1-x^2}}\,\frac{(u^2,1/u^2,\alpha\beta,\alpha\gamma,\alpha\delta,\beta\gamma,\beta\delta,\gamma\delta,q;q)_\infty}{(\alpha u,\alpha/u,\beta u,\beta/u,\gamma u,\gamma/u,\delta u,\delta/u,\alpha\beta\gamma\delta;q)_\infty}, \tag{4.6}$$

normalized by

$$\int_{-1}^{1}w(x)\,dx=1,\qquad |\alpha|<1,\ |\beta|<1,\ |\gamma|<1,\ |\delta|<1.$$

The system of biorthogonal rational functions recently given by Al-Salam and Ismail [1] can be related to the above Askey–Wilson case through a change in parameterization. This will give us our first explicit ${}_4\phi_3$ example of an R_{II}-fraction.

EXAMPLE 4.1 (Biorthogonality on the unit circle). Let us use the new parameters a, b, t_1, t_2 and the variable z given by the replacements

$$\begin{gathered} \alpha = q^{-1/4} t_2 \sqrt{z}, \qquad \beta = q^{3/4} a \sqrt{z}, \qquad \gamma = b q^{1/4}/\sqrt{z}, \\ \delta = q^{-3/4} t_1/\sqrt{z}, \qquad u = q^{-1/4}\sqrt{z}. \end{gathered} \tag{4.7}$$

Then after renormalization (4.3) becomes a recurrence relation of type R_{II}, namely,

$$P_n(z) - (z - c_n) P_{n-1}(z) + \lambda_n (z - a_n)(z - b_n) P_{n-2}(z) = 0 \tag{4.8}$$

with

$$\lambda_{n+1} = \frac{\begin{pmatrix} a t_2 q^n (q^n - 1)(1 - a b t_1 t_2 q^{n-2})(1 - b t_2 q^{n-1})(1 - t_1 t_2 q^{n-2}) \\ \times (1 - a b q^n)(1 - a t_1 q^{n-1}) \end{pmatrix}}{\begin{pmatrix} 4\sqrt{q}\,(1 - a b t_1 t_2 q^{2n-3})(1 - a b t_1 t_2 q^{2n-2})^2 \\ \times (1 - a b t_1 t_2 q^{2n-1})\, u_n u_{n+1} \end{pmatrix}}$$

$$a_{n+1} = b t_1 q^{n-3/2}, \qquad b_{n+1} = q^{-n+1/2}/(a t_2), \qquad c_n = -v_n/u_n,$$

$$\begin{aligned} u_{n+1} = {} & \frac{q^{-1/4}}{2(1 - a b t_1 t_2 q^{2n-2})(1 - a b t_1 t_2 q^{2n})} \\ & \times \left\{ 1 - q^{n-1} \left[\left(1 + \frac{a b t_1 t_2}{q} q^{2n}\right)(q t_2 + a q^2 + a t_1 t_2 + a b t_2 q) \right.\right. \\ & \left.\left. - \frac{a b t_1 t_2}{q} q^n (1 + q)\left(t_2 + q a + \frac{q}{b} + \frac{q^2}{t_1}\right)\right]\right\} \end{aligned}$$

$$\begin{aligned} v_{n+1} = {} & \frac{q^{1/4}}{2(1 - a b t_1 t_2 q^{2n-2})(1 - a b t_1 t_2 q^{2n})} \\ & \times \left\{ 1 - q^{n-1} \left[\left(1 + \frac{a b t_1 t_2}{q} q^{2n}\right)\left(b q + t_1 + a b t_1 + \frac{b t_1 t_2}{q}\right) \right.\right. \\ & \left.\left. - \frac{a b t_1 t_2}{q} q^n (1 + q)\left(b + \frac{t_1}{q} + \frac{q}{t_2} + \frac{1}{a}\right)\right]\right\}. \end{aligned} \tag{4.9}$$

Note that, in order to calculate the middle coefficient in (4.8) and arrive at values for c_n, u_n, and v_n, it is very useful to use the form of the

Askey–Wilson three term recurrence relation given in [5, (1.24), (1.27)]. This is true also for the next example.

The polynomials $\{P_n(z)\}$ are now given by

$$P_n(z) = \frac{q^{n/4}(at_2zq^{1/2}, bt_2, t_1t_2/q, abt_1t_2/q; q)_n}{(2t_2)^n(abt_1t_2/q; q)_{2n}\prod_{k=1}^n u_k} \times {}_4\phi_3\left(\begin{matrix} q^{-n}, abt_1t_2q^{n-1}, t_2zq^{-1/2}, t_2 \\ at_2zq^{1/2}, bt_2, t_1t_2/q \end{matrix} \middle| q, q\right). \quad (4.10)$$

From (4.4) with a renormalization factor $z^{n/2}/\prod_{k=-1}^n u_k$ we obtain the minimal solution to (4.8)

$$X_n^{(\min)}(z) = \begin{cases} X_n^{(1)}(z), & |z| > |q|^{1/2} \\ X_n^{(2)}(z), & |z| < |q|^{1/2}, \end{cases} \quad (4.11)$$

with

$$X_n^{(1)}(z) = \frac{\left(\begin{matrix}(q^{1/4}/2)^n(abt_1t_2q^{2n-1}, t_2q^{n+1}, aq^{n+2}, \\ bq^{n+3/2}/z, t_1q^{n+1/2}/z; q)_\infty\end{matrix}\right)}{\left(\begin{matrix}(q^{n+1}, q^{n+5/2}/z, at_2q^{n+1/2}z, bt_2q^n, t_1t_2q^{n-1}, \\ abq^{n+1}, at_1q^n, bt_1q^{n-1/2}/z; q)_\infty\end{matrix}\right)} \times {}_8W_7\left(\frac{q^{n+3/2}}{z}; q^{n+1}, \frac{q^{3/2}}{t_2z}, \frac{q^{1/2}}{az}, \frac{q}{b}, \frac{q^2}{t_1}; abt_1t_2q^{n-1}\right) \Big/ \prod_{k=-1}^{n} u_k, \quad (4.12)$$

and

$$X_n^{(2)}(z) = \frac{(q^{-1/4}z/2)^n(abt_1t_2q^{2n-1}, t_2zq^{n+1/2}, azq^{n+3/2}, bq^{n+1}, t_1q^n; q)_\infty}{\left(\begin{matrix}(q^{n+1}, zq^{n+3/2}, at_2zq^{n+1/2}, bt_2q^n, t_1t_2q^{n-1}, abq^{n+1}, \\ at_1q^n, bt_1q^{n-1/2}/z; q)_\infty\end{matrix}\right)} \times {}_8W_7\left(q^{n+1/2}z; q^{n+1}, \frac{q}{t_2}, \frac{1}{a}, \frac{q^{1/2}z}{b}, \frac{q^{3/2}z}{t_1}; abt_1t_2q^{n-1}\right) \Big/ \prod_{k=-1}^{n} u_k. \quad (4.13)$$

With the above minimal solution, Eqs. (3.9) and (3.17) imply

$$R_{II}(z) = \frac{2u_1q^{-1/4}(1-abt_1t_2/q)(1-q^{1/2}z)}{(1-b)(1-t_1/q)(1-t_2zq^{-1/2})(1-azq^{1/2})} \times {}_8W_7\left(zq^{1/2}; q, \frac{q}{t_2}, \frac{1}{a}, \frac{q^{1/2}z}{b}, \frac{q^{3/2}z}{t_1}; abt_1t_2/q\right),$$

$$\text{for} \quad |z| < |q|^{1/2}, \quad (4.14)$$

$$R_{II}(z)=\frac{2u_1q^{1/4}(1-abt_1t_2/q)(1-q^{3/2}/z)}{z(1-t_2)(1-aq)(1-bq^{1/2}/z)(1-t_1q^{-1/2}/z)}$$
$$\times {}_8W_7\left(\frac{q^{3/2}}{z};q,\frac{q^{3/2}}{t_2z},\frac{q^{1/2}}{az},\frac{q}{b},\frac{q^2}{t_1};abt_1t_2/q\right),$$
$$\text{for}\quad |z|>|q|^{1/2}. \tag{4.15}$$

An application of Cauchy's theorem gives us the integral representation

$$R_{II}(z)=\int_{|t|=|q|^{1/2}}\frac{\alpha'(t)}{z-t}\,dt, \tag{4.16}$$

with $\alpha'(t)$ as $1/2\pi i$ times the difference of the boundary values of $R_{II}(z)$ in (4.11) as $|z|\to|q|^{1/2}$. The identity [9, (III.37), p. 246] then gives $\alpha'(t)$ explicitly. The result is that $\alpha'(t)$ is $u_1/t^{1/2}$ times the Askey–Wilson weight (4.6) but with the new parameterization given in (4.2) and (4.7), with z replaced by t. In order to isolate the symmetric terms in α' we chose to write it in the form

$$\alpha'(t)=\frac{iu_1}{\pi q^{1/4}}\,\frac{(1-bt_1q^{-1/2}/t)}{(1-t_1/q)(1-b)}f(t), \tag{4.17}$$

where $f(t)$ is

$$f(t)=\frac{(q^{1/2}t,q^{1/2}/t,at_2q^{1/2}t,bt_1q^{1/2}/t,bt_2,at_1,abq,t_1t_2/q,q;q)_\infty}{(aq^{1/2}t,bq^{1/2}/t,aq,bq,t_1,t_2,q^{-1/2}t_2t,q^{-1/2}t_1/t,abt_1t_2;q)_\infty}. \tag{4.18}$$

Note that $f(t)$ is symmetric under the transformation

$$(a,b,t_1,t_2,t)\to(b,a,t_2,t_1,1/t). \tag{4.19}$$

We now derive the biorthogonality relation of Al-Salam and Ismail [1]. In the case under consideration our orthogonality relation (3.15) becomes

$$\frac{iu_1}{\pi q^{1/4}}\int_{|t|=|q|^{1/2}}\frac{t^kP_n(t)(1-bt_1q^{-1/2}/t)f(t)\,dt}{t^n(bt_1q^{-1/2}/t;q)_n\prod_{j=1}^n[t-q^{-j+1/2}/(at_2)]}=0,\qquad 0\leqslant k<n. \tag{4.20}$$

We follow the notation in [1] and define rational functions r_n and s_n by

$$r_n(z)=r_n(z;a,b,t_1,t_2):={}_4\phi_3\left(\begin{matrix}q^{-n},abt_1t_2q^{n-1},q^{-1/2}t_2z,t_2\\ at_2q^{1/2}z,bt_2,t_1t_2/q\end{matrix}\,\middle|\,q,q\right). \tag{4.21}$$

and

$$s_n(z)=s_n(z;a,b,t_1,t_2):=r_n(z;b,a,t_2,t_1). \tag{4.22}$$

There is a slight difference between the above definition of s_n and the definition of s_n in [1], due to complex conjugation. Now the relationship between P_n and r_n is

$$r_n(z;a,b,t_1,t_2)=\frac{C_nP_n(z)}{(at_2q^{1/2}z;q)_n}, \tag{4.23}$$

where the normalization constant C_n is given by

$$C_n=\frac{(2t_2q^{-1/4})^n(abt_1t_2/q;q)_{2n}\prod_{k=1}^n u_k}{(bt_2,t_1t_2/q,abt_1t_2/q;q)_n}.$$

We may now rewrite (4.20) as

$$\int_{|t|=|q|^{1/2}} r_n(t)\,s_k(1/t)f(t)\frac{dt}{t}=0, \qquad k<n. \tag{4.24}$$

If we deform the contour in (4.23) to the unit circle $|t|=1$ and combine this new integral with its complex conjugate (for which $\bar{t}=1/t$) then perform the replacements $(a,b,t_1,t_2,q)\to(\bar{b},\bar{a},\bar{t}_2,\bar{t}_1,\bar{q})$ while taking into account the symmetry of f under the transformation (4.19) we obtain the biorthogonality relation

$$\int_{|t|=1} r_k(t;a,b,t_1,t_2)\,s_n(1/t;a,b,t_1,t_2)f(t)\frac{dt}{t}=0, \qquad k\neq n, \quad n,k\geqslant 0. \tag{4.25}$$

The value of this last integral when $n=k$ is deduced from a modification of (3.16) with $k=n$ and $z=bt_1q^{n-1/2}<1$. Namely,

$$\frac{z^n\tilde{P}_n(1/z)\,X_n^{(\min)}(z)}{\lambda_1[\prod_{j=1}^{n+1}(z-a_j)(z-b_j)]\,X_{-1}^{(\min)}(z)}
=\int_{|t|=|q|^{1/2}}\frac{t^n\tilde{P}_n(1/t)\,P_n(t)\,\alpha'(t)\,dt}{[\prod_{j=2}^{n+1}(t-a_j)(t-b_j)](z-t)}, \tag{4.26}$$

where $\tilde{P}_n(z) = P_n(x; b, a, t_2, t_1)$. We shift the above contour to $|t| = 1$ and put $z = bt_1 q^{n-1/2}$. At this value of z the ${}_8W_7$ in $X_n^{(\min)}(z)$ is summable using [9, III.23, p. 243]. After a lengthy calculation we arrive at

$$\frac{i}{2\pi}\int_{|t|=1} r_n(t)\, s_n(1/t) f(t)\frac{dt}{t} = -\frac{(t_1t_2/q)^n (q, abq, abt_1t_2q^{n-1}; q)_n}{(t_1t_2/q; q)_n (abt_1t_2; q)_{2n}}, \tag{4.27}$$

in agreement with [1].

The condition for the absence of the discrete spectrum and the validity of the integrals on the contour $|t| = |q|^{1/2}$ is $|a|, |b|, |t_1|, |t_2| < 1$. The condition for shifting the contour to $|t| = 1$ is to have $|a|, |b| < |q|^{-1/2}$ and $|t_1|, |t_2| < |q|^{1/2}$. Thus our overall condition for deriving the biorthogonality is $|a|, |b| < 1$ and $|t_1|, |t_2| < |q|^{1/2}$.

EXAMPLE 4.2 (Biorthogonality on the line). We next use a different set of parameters t_1, t_2, t_3, t_4 and variable z. We set

$$\begin{aligned} \alpha &= \sqrt{t_3t_4}\,t_1/q, & \beta &= -qe^{\xi}/\sqrt{t_3t_4}, & \gamma &= qe^{-\xi}/\sqrt{t_3t_4}, \\ \delta &= t_2q^{-2}/\sqrt{t_3t_4}, & u &= -\sqrt{t_3/t_4}, & z &= \sinh\xi. \end{aligned} \tag{4.28}$$

The explicit representation (4.1), after renormalization by $\prod_{k=1}^n u_k$ is now

$$P_n(z; t_1, t_2, t_3, t_4|q) = \frac{(-t_1e^{\xi}, t_1e^{-\xi}, t_1t_2t_3t_4q^{-3}, -t_1t_2q^{-2}; q)_n}{(2t_1\sqrt{t_3t_4/q})^n(-t_1t_2/q^2; q)_{2n}\prod_{k=1}^n u_k} \times {}_4\phi_3\left(\begin{matrix} q^{-n}, -t_1t_2q^{n-2}, -t_1t_3/q, -t_1t_4/q \\ -t_1e^{\xi}, t_1e^{-\xi}, t_1t_2t_3t_4q^{-3} \end{matrix}\,\middle|\, q, q\right), \tag{4.29}$$

where u_n is given below in (4.33).

The three term recurrence relation (4.3), after renormalization is now of type R_{II} and is

$$P_n(z) - (z - c_n)\,P_{n-1}(z) + \lambda_n(z - a_n)(z - b_n)\,P_{n-2}(z) = 0 \tag{4.30}$$

with

$$\lambda_{n+1} = \frac{t_1t_2q^{2n-3}(1-q^n)(1+t_1t_2q^{n-3})(1+q^{n+1}/t_3t_4)(1-t_1t_2t_3t_4q^{n-4})}{(1+t_1t_2q^{2n-2})(1+t_1t_2q^{2n-3})^2(1+t_1t_2q^{2n-4})\,u_nu_{n+1}} \tag{4.31}$$

$$a_{n+1} = \tfrac{1}{2}(t_2q^{n-2} - q^{2-n}/t_2), \qquad b_{n+1} = \tfrac{1}{2}(t_1q^{n-1} - q^{-n-1}/t_1), \tag{4.32}$$

$$c_{n+1} = \frac{-v_{n+1} + (\sqrt{t_3/t_4} + \sqrt{t_4/t_3})/2}{u_{n+1}},$$

where u_n and v_n are given as

$$u_{n+1} = [(1-t_1t_2q^{2n-2})(q^4+t_1t_2t_3t_4)+q^n(1+q)\,t_1t_2(q-t_3t_4)] \times \frac{q^{n-3}/\sqrt{t_3t_4}}{(1+t_1t_2q^{2n-3})(1+t_1t_2q^{2n-1})} \tag{4.33}$$

$$v_{n+1} = -[(1-t_1t_2q^{2n-1})(t_3t_4-q^2)+q^{n-3}(1+q)(t_1t_2t_3t_4+q^4)] \times \frac{q^{n-1}(t_1+t_2/q)}{2\sqrt{t_3t_4}\,(1+t_1t_2q^{2n-3})(1+t_1t_2q^{2n-1})}. \tag{4.34}$$

In the case under consideration the continued fraction is the R_{II} fraction

$$\frac{1}{z-c_1-}\;\frac{\lambda_2(z-a_2)(z-b_2)}{z-c_2}\;\frac{}{-}\cdots. \tag{4.35}$$

When the continued fraction converges it will converge to $F(z)$, where

$$F(z) = -\frac{2u_1}{\sqrt{t_3/t_4}}\,\frac{(1-qt_4/t_3)(1+t_1t_2/q^2)}{(qt_4/t_3,\ -t_1t_4/q,\ qe^{\xi}/t_3,\ -qe^{-\xi}/t_3,\ -t_2t_4/q^2;\ q)_\infty} \times \tilde{W}(qt_4/t_3;\ q,\ -q^2/t_1t_3,\ t_4e^{-\xi},\ -t_4e^{\xi},\ -q^3/t_2t_3;\ t_1t_2/q^2), \tag{4.36}$$

with

$$\tilde{W}\left(a;b,c,d,e,f;\frac{a^2q^2}{bcdef}\right) := (b,c,d,e,f;q)_\infty\ {}_8W_7\left(a;b,c,d,e,f;\frac{a^2q^2}{bcdef}\right).$$

The singularities of (4.36) are at $z = z_n$,

$$z_n = \tfrac{1}{2}(t_3q^{-n-1}-q^{n+1}/t_3), \qquad n = 0, 1, \cdots,$$

given by the zeros of $(qe^{\xi}/t_3, -qe^{-\xi}/t_3; q)_\infty$. The residues at these points may be calculated explicitly because the contribution from $\tilde{W}$ is in terms of a very well-poised ${}_6\phi_5$, which is summable. After a straightforward but rather lengthy calculation, we arrive at the Mittag–Leffler expansion

$$F(z) = \sum_{k=0}^{\infty} \frac{\omega_k}{z-z_k} \tag{4.37}$$

with

$$\omega_k = \frac{u_1\sqrt{t_3 t_4}}{q^{4k+1}} \frac{(t_1 t_2 t_3 t_4 q^{-3}, qt_1/t_3, t_2/t_3, qt_4/t_3; q)_\infty}{(-t_1 t_2/q, -t_1 t_4/q, -t_2 t_4/q^2, -q^2/t_3^2; q)_\infty}$$
$$\times \frac{(-q^2/t_1 t_3, -q^3/t_2 t_3, -q^2/t_3 t_4, -q^2/t_3^2; q)_k}{(qt_1/t_3, t_2/t_3, qt_4/t_3, q; q)_k} (1+q^{2k+2}/t_3^2)(t_1 t_2 t_3 t_4)^k. \tag{4.38}$$

The resulting orthogonality (3.15) is now given by

$$\sum_{k=0}^{\infty} \frac{z_k^m P_n(z_k)\,\omega_k}{\prod_{j=1}^n (z_k - a_{j+1})(z_k - b_{j+1})} = 0, \qquad 0 \leqslant m < n. \tag{4.39}$$

We now derive a rational biorthogonality relation. Let us include the factor

$$z_k - a_2 = \frac{t_3}{2} q^{-k-1}(1 - t_2 q^k/t_3)(1 + q^{k+2}/t_2 t_3)$$

in the weight function to obtain the new weight function

$$r_k = \frac{\omega_k}{z_k - a_2}$$
$$= \frac{2t_2 u_1 \sqrt{t_3 t_4}}{q^{3k}(q^2 + t_2 t_3)} \frac{(t_1 t_2 t_3 t_4 q^{-3}, qt_1/t_3, qt_2/t_3, qt_4/t_3; q)_\infty}{(-t_1 t_2/q, -t_1 t_4/q^2, -t_2 t_4/q^2, -q^2/t_3^2; q)_\infty}$$
$$\times \frac{(-q^2/t_1 t_3, -q^2/t_2 t_3, -q^2/t_3 t_4, -q^2/t_3^2; q)_k}{(qt_1/t_3, qt_2/t_3, qt_4/t_3, q; q)_k} (1+q^{2k+2}/t_3^2)(t_1 t_2 t_3 t_4)^k.$$

Note that this has k dependent terms which are now symmetric in t_1 and t_2. This symmetry together with (4.39) yields the biorthogonality relation

$$\sum_{k=0}^{\infty} \tilde{R}_m(z_k)\, R_n(z_k)\, r_k = 0, \qquad m \neq n, \tag{4.40}$$

where

$$R_n(z) = R_n(z; t_1, t_2, t_3, t_4)$$
$$= {}_4\phi_3\left(\begin{matrix} q^{-n}, -t_1 t_2 q^{n-2}, -t_1 t_3/q, -t_1 t_4/q \\ -t_1 e^{\xi}, t_1 e^{-\xi}, t_1 t_2 t_3 t_4 q^{-3} \end{matrix} \,\middle|\, q, q\right), \tag{4.41}$$
$$\tilde{R}_n(z) = R_n(z; t_2, t_1, t_3, t_4),$$

and from (4.29) we have

$$\frac{P_n(z)}{\prod_{j=1}^n (z-b_{j+1})} = (-1)^n q^{n(n+1)/2} (t_3t_4)^{-n/2} \frac{(-t_1t_2/q^2, t_1t_2t_3t_4q^{-3}; q)_n}{(-t_1t_2/q^2; q)_{2n} \prod_{k=1}^n u_k} R_n(z). \tag{4.42}$$

For the case $m=n$ we use (3.16) with $k=n$ and z^n replaced by $P_n(z; t_2, t_1, t_3, t_4)$ and $z=a_{n+2}$. This requires the evaluations

$$P_n(a_{n+2}; t_2, t_1, t_3, t_4) = \frac{q^{-n(n-3)/2}(-t_2t_3/q, -t_2t_4/q; q)_n}{(-2t_2\sqrt{t_3t_4})^n \prod_{k=1}^n u_k} \tag{4.43}$$

and from [9, (III.24)]

$$\begin{aligned} &{}_8W_7(q^{n+1}t_4/t_3; q^{n+1}, -q^2/t_1t_3, t_4e^{-\xi}, -t_4e^{\xi}, -q^3/t_2t_3; t_1t_2/q)|_{e^{\xi}=t_2q^{n-1}} \\ &= \frac{(q^{n+2}t_4/t_3, -t_1t_4/q, q^nt_2/t_3, -q^{2n-1}t_1t_2; q)_\infty}{(q^{2n+1}t_2/t_3, -q^nt_1t_4, qt_4/t_3, -t_1t_2q^{n-2}; q)_\infty}. \end{aligned} \tag{4.44}$$

The final result, after another lengthy calculation, is

$$\begin{aligned} &\sum_{k=0}^{\infty} R_m(z_k) \tilde{R}_n(z_k) \frac{(-q^2/t_1t_3, -q^2/t_2t_3, -q^2/t_3t_4, -q^2/t_3^2; q)_k}{(qt_4/t_3, qt_1/t_3, qt_2/t_3, q; q)_k} \\ &\quad \times \frac{1+q^{2k+2}/t_3^2}{1+q^2/t_3^2} (t_1t_2t_3t_4/q^3)^k \\ &= (t_1t_2t_3t_4/q^3)^n \frac{1+t_1t_2/q^2}{1+t_1t_2q^{2n-2}} \frac{(-q^2/t_3t_4, q; q)_n}{(-t_1t_2/q^2, t_1t_2t_3t_4/q^3; q)_n} \\ &\quad \times \frac{(-t_1t_2/q, -t_1t_4/q, -t_2t_4/q, -q^3/t_3^2; q)_\infty}{(qt_1/t_3, qt_2/t_3, qt_4/t_3, t_1t_2t_3t_4/q^3; q)_\infty} \delta_{m,n}, \end{aligned} \tag{4.45}$$

with

$$z_k = \tfrac{1}{2}(t_3q^{-k-1} - q^{k+1}/t_3), \qquad k=0, 1, ..., \qquad |t_1t_2t_3t_4| < q^3.$$

Another biorthogonality relation can be obtained by interchanging t_3 and t_4, which corresponds to taking $u = -\sqrt{t_4/t_3}$. Both of these are special cases of the biorthogonality relations derived in [18] (see (1.21), (3.14), and (3.15) in [18]), with $a=q/t_3$ or $a=q/t_4$. However, if in (4.28) we had used $\beta = qe^{\xi}/\sqrt{t_3 t_4}$ and $z = \cosh \xi$, then we would have arrived at a similar biorthogonality with mass points at

$$z_k = \tfrac{1}{2}(t_3 q^{-k-1} + q^{k+1}/t_3), \qquad k = 0, 1, \ldots.$$

We do not know where this fits into the scheme of things other than to suggest that, with discrete orthogonalities at this level, there are always two families, one associated with $z = \sinh \xi$, and another with $z = \cosh \xi$.

5. Biorthogonality on $[-1, 1]$

We present two final examples by yet another modification of the Askey–Wilson recurrence relation. Here an R_{II}-type continued fraction is used to obtain two q-beta integrals, one which is new and one which is in [27]. However, the biorthogonalities obtained are not given by the continued fraction methods of the previous examples. Instead we use the "attachment" method explained in [6] which goes back to Andrews and Askey and was used in [5, 1].

In (4.3) we apply the parameter replacements

$$\beta \to \beta/u, \qquad \gamma \to \beta u, \tag{5.1}$$

with $z = (u + 1/u)/2$. The recurrence relation again becomes an R_{II}-type recurrence relation (4.30) but now with

$$\lambda_{n+1} = \frac{\alpha\beta^2\delta q^{2n-2}(1-q^n)(1-\alpha\beta^2\delta q^{n-2})(1-\alpha\delta q^{n-1})(1-\beta^2 q^{n-1})}{(1-\alpha\beta^2\delta q^{2n-3})(1-\alpha\beta^2\delta q^{2n-2})^2(1-\alpha\beta^2\delta q^{2n-1})\, u_n u_{n+1}}, \tag{5.2}$$

$$a_{n+1} = \tfrac{1}{2}(\beta\delta q^{n-1} + q^{1-n}/\beta\delta), \qquad b_{n+1} = \tfrac{1}{2}(\alpha\beta q^{n-1} + q^{1-n}/\alpha\beta), \tag{5.3}$$

$$c_{n+1} = v_{n+1}/u_{n+1}, \tag{5.4}$$

where u_n and v_n are given by

$$u_{n+1} = 1 - \beta q^{n-1} \frac{(1+\alpha\beta^2\delta q^{2n-1})(q+\alpha\delta) - q^{n-1}(1+q)(q+\beta^2)\,\alpha\delta}{(1-\alpha\beta^2\delta q^{2n-2})(1-\alpha\beta^2\delta q^{2n})}, \tag{5.5}$$

$$v_{n+1} = (\alpha+\delta)\, q^{n-1} \frac{(1+\alpha\beta^2\delta q^{2n-1})(q+\beta^2) - q^{n-1}(1+q)\,\beta^2(q+\alpha\delta)}{2(1-\alpha\beta^2\delta q^{2n-2})\, 1-\alpha\beta^2\delta q^{2n})}. \tag{5.6}$$

The new minimal solution of the recurrence relation is given by

$$X_n^{(\min)}(z) = \begin{cases} G_n(u), & |u| > 1 \\ G_n(1/u), & |u| < 1, \end{cases} \tag{5.7}$$

$$G_n(u) = \frac{(2u)^{-n}(\alpha\beta^2\delta q^{2n-1}, \alpha q^{n+1}/u, \beta q^{n+1}/u^2, \beta q^{n+1}, \delta q^{n+1}/u; q)_\infty}{(q^{n+1}, q^{n+2}/u^2, \alpha\beta q^n u, \alpha\beta q^n/u, \alpha\delta q^n, \beta^2 q^n, \beta\delta q^n u, \beta\delta q^n/u; q)_\infty}$$

$$\times {}_8W_7(q^{n+1}/u^2; q^{n+1}, q/\alpha u, q/\beta, qu^{-2}/\beta, q/\delta u; \alpha\beta^2\delta q^{n-1}) \Big/ \prod_{k=-1}^{n} u_k. \tag{5.8}$$

An application of Pincherle's theorem now yields the Stieltjes transform

$$\int_{-1}^{1} \frac{f(x)\,dx}{z-x} = \frac{2}{u} \frac{(1-q/u^2)(1-\alpha\beta^2\delta/q)}{(1-\alpha/u)(1-\beta/u^2)(1-\beta)(1-\delta/u)}$$

$$\times {}_8W_7(q/u^2; q, q/\alpha u, q/\beta, q/\beta u^2, q/\delta u; \alpha\beta^2\delta/q), \tag{5.9}$$

with $z = \frac{1}{2}(u + 1/u)$, $|u| > 1$, and

$$f(\cos\theta) := \frac{1}{2\pi} \frac{(e^{2i\theta}, e^{-2i\theta}, \alpha\beta e^{i\theta}, \alpha\beta e^{-i\theta}, \beta\delta e^{i\theta}, \beta\delta e^{-i\theta}, \alpha\delta, \beta^2, q; q)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, \beta e^{2i\theta}, \beta e^{-2i\theta}, \delta e^{i\theta}, \delta e^{-i\theta}, \beta, \beta, \alpha\beta^2\delta; q)_\infty} \frac{1}{\sin\theta}. \tag{5.10}$$

The integral (5.9) seems to be new.

The large z asymptotics of the transform formula (5.9) produces the following result

$$\frac{1}{2\pi}\int_0^\pi f(\cos\theta)\sin\theta\,d\theta = \frac{1-\alpha\beta^2\delta/q}{1-\beta}\, {}_2\phi_1(q, q/\beta; q\beta; q, \alpha\beta^2\delta/q), \tag{5.11}$$

valid for $\max\{|\alpha|, |\beta|, |\delta|, |\alpha\beta^2\delta/q|\} < 1$.

We next explain where (5.11) comes from in the theory of orthogonal polynomials. Recall that the continuous q-ultraspherical polynomials $\{C_n(x; \beta|q)\}$ have the generating function

$$\sum_{n=0}^{\infty} C_n(\cos\theta; \beta \mid q)\, t^n = \frac{(\beta t e^{i\theta}, \beta t e^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} \tag{5.12}$$

and satisfy the orthogonality relation

$$\frac{1}{2\pi}\int_0^\pi C_m(\cos\theta; \beta \mid q)\, C_n(\cos\theta; \beta \mid q) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty}\, d\theta$$

$$= \frac{(\beta, q\beta; q)_\infty (\beta^2; q)_n (1-\beta)}{(q, \beta^2; q)_\infty (q; q)_n (1-\beta q^n)} \delta_{m,n} \tag{5.13}$$

[3, 9]. It is easy to see that (5.12) and (5.13) show that the left-hand side of (5.11) is

$$= \sum_{m,n=0}^{\infty} \frac{\alpha^n \delta^m}{2\pi} \int_0^{\pi} C_m(\cos\theta; \beta \mid q)\, C_n(\cos\theta; \beta \mid q) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty}\, d\theta$$

$$= \sum_{n=0}^{\infty} (\alpha\delta)^n \frac{(\beta, q\beta; q)_\infty (\beta^2; q)_n (1-\beta)}{(q, \beta^2; q)_\infty (q; q)_n (1-\beta q^n)}$$

$$= \frac{(\beta, q\beta; q)_\infty}{(q, \beta^2; q)_\infty}\, {}_2\phi_1(\beta^2, \beta; q\beta; q, \alpha\delta).$$

Now the Heine transformation [9, (III.3)] reduces the extreme right-hand side above to the right-hand side of (5.11).

EXAMPLE 5.1 (Chebyshev rational functions). The relationship (5.11) gives an integral representation for a basic hypergeometric function of the type ${}_2\phi_1$. In this generality the integral representation (5.11) does not seem to lead to orthogonal or biorthogonal functions. The special case $\beta = q$ degenerates into the elementary integral result (1.3). To find rational functions biorthogonal with respect to the integrand in (1.3) we now use the attachment method.

Consider functions of the type

$$g_n(\cos\theta; \alpha, \delta) := \sum_{k=0}^{n} \frac{(q^{-n}, \alpha e^{i\theta}, \alpha e^{-i\theta}; q)_k}{(q, q\alpha e^{i\theta}, q\alpha e^{-i\theta}; q)_k} a_{n,k}, \tag{5.14}$$

where $a_{n,k}$ are to be determined. Set

$$I_{n,j} = \frac{2}{\pi} \int_0^{\pi} g_n(\cos\theta; \alpha, \delta) \frac{(\delta e^{i\theta}, \delta e^{-i\theta}; q)_j}{(q\delta e^{i\theta}, q\delta e^{-i\theta}; q)_j} \times \frac{\sin^2\theta\, d\theta}{(1 - 2\alpha\cos\theta + \alpha^2)(1 - 2\delta\cos\theta + \delta^2)}. \tag{5.15}$$

Thus

$$I_{n,j} = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} a_{n,k} \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}\, dx}{(1 - 2\alpha q^k x + \alpha^2 q^{2k})(1 - 2\delta q^j x + \delta^2 q^{2j})}$$

$$= \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} a_{n,k} \frac{1}{1 - \alpha\delta q^{k+j}}$$

$$= \frac{1}{1 - \alpha\delta q^j} \sum_{k=0}^{n} \frac{(q^{-n}, \alpha\delta q^j; q)_k}{(q, \alpha\delta q^{j+1}; q)_k} a_{n,k}.$$

Recall that the q-analogue of the Pfaff–Saalschütz theorem [9, II.12)] is

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, a, b \\ c, abq^{1-n}/c \end{matrix} \middle| q, q\right) = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}. \tag{5.16} $$

We would like to choose $a_{n,k}$ to make $I_{n,j}$ vanish for $0 \leqslant j < n$. In view of (5.16) we let $a_{n,k} = q^k(\alpha\delta q^n; q)_k/(\alpha\delta; q)_k$ and we obtain

$$ I_{n,j} = \frac{(q^{-n}, q^{-j}; q)_n}{(1-\alpha\delta q^j)(\alpha\delta, q^{-n-j}/\alpha\delta; q)_n}. $$

It is now clear that $I_{n,j} = 0$ if $0 \leqslant j < n$. After some simplification we find

$$ I_{n,j} = [(-\alpha\delta)^n q^{n(n-1)/2}(q; q)_n^2/(\alpha\delta; q)_{2n+1}]\, \delta_{j,n}. $$

Thus the g_n's of (5.14) are given by

$$ g_n(\cos\theta; \alpha, \delta) = {}_4\phi_3\left(\begin{matrix} q^{-n}, \alpha\delta q^n, \alpha e^{i\theta}, \alpha e^{-i\theta} \\ \alpha\delta, q\alpha e^{i\theta}, q\alpha e^{-i\theta} \end{matrix} \middle| q, q\right) \tag{5.17} $$

and satisfy the orthogonality relation

$$ \frac{2}{\pi}\int_{-1}^{1} g_m(x; \delta, \alpha)\, g_n(x; \alpha, \delta) \frac{\sqrt{1-x^2}\,dx}{(1-2\alpha x+\alpha^2)(1-2\delta x+\delta^2)} $$
$$ = \frac{(\alpha\delta)^n(q, q; q)_n}{(\alpha\delta, \alpha\delta; q)_n(1-\alpha\delta q^{2n})}\delta_{m,n}. \tag{5.18} $$

We find the biorthogonality relation (5.18) very surprising since the ${}_4\phi_3$ function in (5.17) is not balanced. Recall that a basic hypergeometric function (1.9) is balanced if $r = s+1$ and $qa_1a_2 \cdots a_{s+1} = b_1b_2 \cdots b_s$. Only balanced ${}_4\phi_3$'s with argument q satisfy three term recurrence relations. This concludes this example.

It is worth noting that the special case $q = 0$ of the Askey–Wilson polynomials gives rise to polynomials orthogonal with respect to the weight function [5]

$$ \frac{1}{(1-2ax+a^2)(1-2bx+b^2)(1-2cx+c^2)(1-2dx+d^2)}. $$

The polynomials are linear combinations of Chebyshev polynomials. The rational functions biorthogonal with respect to the above weight function are still under investigation.

In the previous examples of pairs of biorthogonal functions the second family was obtained from the first by symmetry and permutation of parameters. In the next example the members of the pair of biorthogonal rational functions are not related in this manner.

EXAMPLE 5.2 (A nonsymmetric case). Following previous examples we now consider the q-beta integral obtained by absorbing the factor $x-a_2$ into the weight function. This means evaluating (5.9) at $z=a_2$, that is $u=1/\alpha\beta$, and this means evaluating (5.9) at $z=a_2$, that is $u=1/\alpha\beta$, and this is exactly the case that makes the right-hand side of (5.9) summable. This gives the result

$$\frac{1}{2\pi}\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}, \alpha\beta q e^{i\theta}, \alpha\beta q e^{-i\theta}, \beta\delta e^{i\theta}, \beta\delta e^{-i\theta}, \alpha\delta, \beta^2, q; q)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, \beta e^{2i\theta}, \beta e^{-2i\theta}, \delta e^{i\theta}, \delta e^{-i\theta}, \beta, q\beta, \alpha\beta^2\delta; q)_\infty}\, d\theta$$

$$=\frac{1}{1-\alpha^2\beta}, \qquad \max\{|\alpha|, |\beta|, |\delta|\}<1. \tag{5.19}$$

After we showed an earlier version of this paper to Mizan Rahman he pointed out that this q-beta integral is not new. It was obtained previously by him in his interesting work [27] using a different method. It may be of interest to mention here that (5.19) is also the special case $\gamma=q\beta$ of

$$\frac{1}{2\pi}\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}, \alpha\gamma e^{i\theta}, \alpha\gamma e^{-i\theta}, \beta\delta e^{i\theta}, \beta\delta e^{-i\theta}, \alpha\delta, \beta^2, q; q)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, \beta e^{2i\theta}, \beta e^{-2i\theta}, \delta e^{i\theta}, \delta e^{-i\theta}, \beta, q\beta, \alpha\beta^2\delta; q)_\infty}\, d\theta$$

$$=\frac{(\beta, \gamma, \alpha^2\gamma, \beta^2\alpha\delta; q)_\infty}{(q, \beta^2, \alpha^2\beta, \alpha\delta; q)_\infty}\, {}_3\phi_2\left(\begin{matrix}\alpha^2\beta, \alpha\delta, q\beta/\gamma \\ \alpha^2\gamma, \beta^2\alpha\delta\end{matrix}\,\middle|\, q, \gamma\right) \tag{5.20}$$

for $\max\{|\alpha|, |\beta|, |\gamma|, |\delta|\}<1$, which will appear elsewhere. In view of (5.12) and (5.13) Eq. (5.20) is equivalent to the evaluation of the connection coefficients for the continuous q-ultraspherical polynomials. Rogers solved this connection coefficient problem in 1893; see [3].

One can use the attachment method of the previous example to derive a system biorthogonal with respect to the integrand in (5.19). Rahman also has done this in [27]. For completeness we repeat the calculation here.

The q-beta integral we are working with is

$$\int_{-1}^{1} w(x; \alpha, \beta, \delta)\, dx = \frac{(\beta, q\beta, \alpha\beta^2\delta; q)_\infty}{(1-\alpha^2\beta)(\alpha\delta, \beta^2, q; q)_\infty}, \tag{5.21}$$

where

$$w(\cos\theta; \alpha, \beta, \delta) := \frac{(e^{2i\theta}, e^{-2i\theta}, q\alpha\beta e^{i\theta}, q\alpha\beta e^{-i\theta}, \beta\delta e^{i\theta}, \beta\delta e^{-i\theta}; q)_\infty}{2\pi \sin\theta(\alpha e^{i\theta}, \alpha e^{-i\theta}, \beta e^{2i\theta}, \beta e^{-2i\theta}, \delta e^{i\theta}, \delta e^{-i\theta}; q)_\infty}. \tag{5.22}$$

Choose

$$\psi_n(\cos\theta;\alpha,\beta,\delta)=\sum_{k=0}^{n}\frac{(q^{-n},\delta e^{i\theta},\delta e^{-i\theta};q)_k}{(q,\delta\beta e^{i\theta},\delta\beta e^{-i\theta};q)_k}a_{n,k}. \tag{5.23}$$

Using the q-analogue of the Pfaff–Saalschütz theorem (5.16), we put

$$a_{n,k}=q^k\frac{(\alpha\beta^2\delta q^{n-1};q)_k}{(\alpha\delta;q)_k}$$

and find

$$\int_0^{\pi} w(\cos\theta;\alpha,\beta,\delta)\,\psi_n(\cos\theta;\alpha,\beta,\delta)\frac{(\alpha e^{i\theta},\alpha e^{-i\theta};q)_j}{(q\alpha\beta e^{i\theta},q\alpha\beta e^{-i\theta};q)_j}\sin\theta\,d\theta.$$

$$=\frac{(\beta,q\beta,\alpha\beta^2q^n\delta;q)_\infty}{(1-\alpha^2\beta q^{2n})(\alpha\delta q^n,\beta^2,q;q)_\infty}\,\frac{(\beta^2,q;q)_n}{(\alpha\beta^2\delta q^n,q^{1-n}/\alpha\delta;q)_n}\delta_{j,n},\qquad j\leqslant n. \tag{5.24}$$

Let

$$\psi_n(\cos\theta;\alpha,\beta,\delta)={}_4\phi_3\left(\begin{matrix}q^{-n},\alpha\beta^2\delta q^{n-1},\delta e^{i\theta},\delta e^{-i\theta}\\ \alpha\delta,\beta\delta e^{i\theta},\beta\delta e^{-i\theta}\end{matrix}\,\middle|\,q,q\right) \tag{5.25}$$

and

$$\phi_n(\cos\theta;\alpha,\beta,\delta):=\sum_{j=0}^{n}\frac{(q^{-n},\alpha e^{i\theta},\alpha e^{-i\theta};q)_j}{(q,q\alpha\beta e^{i\theta},q\alpha\beta e^{-i\theta};q)_j}q^jb_{n,j}. \tag{5.26}$$

Thus we have proved that the rational functions $\{\phi_n\}$ and $\{\psi_n\}$ satisfy the biorthogonality relation

$$\int_{-1}^{1}\phi_m(x;\alpha,\beta,\delta)\,\psi_n(x;\alpha,\beta,\delta)\,w(x;\alpha,\beta,\delta)\,dx$$

$$=\frac{(\beta,q\beta,\alpha\beta^2\delta q^n;q)_\infty}{(\alpha\delta,\beta^2,q;q)_\infty}\,\frac{(\beta^2,q;q)_n}{(1-\alpha^2\beta q^n)}(-\alpha\delta)^n\,b_{n,n}\delta_{m,n},\qquad m\leqslant n. \tag{5.27}$$

We now select $b_{n,j}$ in order to have full biorthogonality, that is to make (5.27) hold for all m and n. In order to do so we need to compute the integrals $J_{n,k}$,

$$J_{n,k}:=\int_0^{\pi}\phi_n(\cos\theta;\alpha,\beta,\delta)\frac{(\delta e^{i\theta},\delta e^{-i\theta};q)_k}{(\beta\delta e^{i\theta},\beta\delta e^{-i\theta};q)_k}w(\cos\theta;\alpha,\beta,\delta)\sin\theta\,d\theta$$

for $k \leqslant n$. Substituting for ϕ_n from (5.26) and using the q-beta integral (5.21) we get

$$J_{n,k} = \frac{(\beta, q\beta, \alpha\beta^2\delta q^k; q)_\infty}{(\beta^2, \alpha\delta q^k; q)_\infty} \sum_{j=0}^{n} \frac{(q^{-n}, \alpha\delta q^k; q)_j}{q, \alpha\beta^2\delta q^k; q)_j} \frac{q^j b_{n,j}}{1-\alpha^2\beta q^{2j}}.$$

In order to apply (5.16) we must choose $b_{n,j}$ as

$$b_{n,j} = \frac{(1-\alpha^2\beta q^{2j})(\alpha\beta^2\delta q^{n-1}; q)_j}{(1-\alpha^2\beta)(\alpha\delta; q)_j}; \tag{5.28}$$

hence

$$J_{n,k} = \frac{(\beta, q\beta, \alpha\beta^2\delta q^k; q)_\infty}{(\beta^2, \alpha\delta q^k; q)_\infty} \frac{(q^{-k}, q^{1-n}/\beta^2; q)_n}{(\alpha\delta, q^{1-k-n}/\alpha\beta^2\delta; q)_\infty}.$$

Thus (5.26) becomes

$$\phi_n(\cos\theta, \alpha, \beta, \delta) = {}_6\phi_5\left(\begin{matrix} q^{-n}, \alpha\beta^2\delta q^{n-1}, \alpha e^{i\theta}, \alpha e^{-i\theta}, q\alpha\sqrt{\beta}, -q\alpha\sqrt{\beta} \\ \alpha\delta, q\alpha\beta e^{i\theta}, q\alpha\beta e^{-i\theta}, \alpha\sqrt{\beta}, -\alpha\sqrt{\beta} \end{matrix} \middle| q, q\right). \tag{5.29}$$

This establishes the biorthogonality relation

$$\int_{-1}^{1} \phi_m(x; \alpha, \beta, \delta)\, \psi_n(x; \alpha, \beta, \delta)\, w(x; \alpha, \beta, \delta)\, dx$$

$$= \frac{(\beta, q\beta, \alpha\beta^2\delta q^n; q)_\infty}{(1-\alpha^2\beta)(\alpha\delta, \beta^2, q; q)_\infty} \frac{(\beta^2, q; q)_n (1-\alpha^2\beta\delta q^{2n-1})(\alpha\delta)^n}{(1-\alpha^2\beta\delta q^{n-1})(\alpha\delta; q)_n} \delta_{m,n}, \tag{5.30}$$

valid for $|\alpha|, |\beta|, |\delta| < 1$.

Special cases of the ψ_n's have appeared in the literature as special cases of the Askey–Wilson polynomials. The case $\beta = 0$ gives the Al-Salam–Chihara polynomials [4, 5], as can be seen by comparing (5.26) and (5.29) with (4.1). The case $\alpha = \delta = 0$ of (5.21), (5.22), (5.26), and (5.29) gives the q-ultraspherical polynomials of Rogers [3, 5] but the justification is not as straightforward. It can be justified, however, by using the generating function [19]

$$\sum_{0}^{\infty} \frac{(ac, ad; q)_n}{(q, cd; q)_n} (t/a)^n \, {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q, q\right)$$

$$= {}_2\phi_1\left(\begin{matrix} ae^{i\theta}, be^{i\theta} \\ ab \end{matrix} \middle| q, q\right) {}_2\phi_1\left(\begin{matrix} ce^{-i\theta}, de^{-i\theta} \\ cd \end{matrix} \middle| q, q\right). \tag{5.31}$$

In a private communication. Askey mentioned that he can let $\alpha \to 0$ in (4.1) and can show directly that $\lim_{\alpha \to 0} \alpha^{-n} P_n(x; \alpha, \beta, \gamma, \delta|q)$ exists and find its value. We do not know how to compute the limit $\lim_{\alpha \to 0} \alpha^{-n} \phi_n(x; \alpha, \beta, \delta|q)$ directly but we suspect that Askey's direct argument may work here. In [27] Rahman made the transition to the $\alpha = \delta = 0$ case by using a ${}_3\phi_2$ representation for the q-ultraspherical polynomials, writing ϕ_n as a combination of ${}_4\phi_3$'s, applying Sear's transformation for ${}_4\phi_3$'s to the ψ_n and the ${}_4\phi_3$'s in ϕ_n, and then taking the limit α, $\delta \to 0$.

Acknowledgments

We thank the referee for his/her careful and critical reading of the first version of this paper and for many helpful suggestions. We also thank our friend Mizan Rahman for comments and for pointing out the overlap of Example 5.2 and his earlier work [27].

Note added in proof. The continued fraction in Example 3.2 is a special case of the one considered by E. L. Ince in E. L. Ince, On the continued fractions connected with the hypergeometric equation, *Proc. Lond. Math. Soc. (2)* **28** (1919), 236–248.

We thank W. B. Jones for bringing this to our attention.

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